A LARCH(∞) Vector Valued Process

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1 Introduction

The purpose of this chapter is to propose a unified framework for the study of ARCH(∞) processes that are commonly used in the financial econometrics literature. We extend the study, based on Volterra expansions, of univariate ARCH(∞) processes by Giraitis *et al.* [GKL00] and Giraitis and Surgailis [GS02] to the multi-dimensional case.

Let $\{\xi_t\}_{t\in\mathbb{Z}}$ be a sequence of real valued random matrices independent and identically distributed of size $d \times m$, $\{a_j\}_{j\in\mathbb{N}^*}$ be a sequence of real matrices $m \times d$, and a be a real vector of dimension m. The vector ARCH(∞) process is defined as the solution to the recurrence equation:

$$X_t = \xi_t \left(a + \sum_{j=1}^{\infty} a_j X_{t-j} \right). \tag{1}$$

The following section 2 displays a chaotic expansion solution to this equation; we also consider a random fields extension of this model. Some approximations of this solutions are listed in the next section 3, where we consider approximations by m-dependent sequences, coupling results and approximations by Markov sequences. Section 4 details the weak dependence properties of the model and section 5 provides an existence and uniqueness condition for the solution of the previous equation; in that case, long range dependence may occur. The end of this section is dedicated to review examples of this vector valued model.

The vector ARCH(∞) model nests a large variety of models, the two first extensions being obvious:

(A1). The univariate linear ARCH(∞) (LARCH) model, where the X_t and a_j are scalar, (A2). The bilinear model, with

$$X_t = \zeta_t \left(\alpha + \sum_{j=1}^{\infty} \alpha_j X_{t-j} \right) + \beta + \sum_{j=1}^{\infty} \beta_j X_{t-j} \ ,$$

where all variables are scalar, and ζ_t are iid centered innovations. We set

$$\xi_t = (\zeta_t, 1) \,, \quad a = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad a_j = \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} \ .$$

In that case, the expansion (3) is the same as the one used by Giraitis and Surgailis [GS02].

(A3). With a suitable re-parameterization, this vector ARCH(∞) nests the standard GARCH–type processes used in the financial econometrics literature for modeling the non-linear structure of the conditional second moments. The GARCH(p, q) model is defined as

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t \ , \\ \sigma_t^2 &= \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 + \gamma_0 + \sum_{j=1}^{q} \gamma_j r_{t-j}^2 \ , \quad \gamma_0 > 0 \ , \quad \gamma_j \geq 0 \ , \quad \beta_i \geq 0 \ , \end{aligned}$$

where the ε_t are centered and iid. This model is nested in the class of bilinear models with the following re-parameterization

$$\alpha_0 = \frac{\gamma_0}{1 - \sum \beta_i} \ , \quad \sum \alpha_i z^i = \frac{\sum \gamma_i z^i}{1 - \sum \beta_i z^i} \ ,$$

see Giraitis *et al.* [GLS05]. The covariance function of the sequence $\{r_t^2\}$ has an exponential decay, which is implied by the exponential decay of the sequence of weights α_j; see Giraitis *et al.* [GKL00].

(A4). The ARCH(∞) model, where the sequence of weights β_j might have either a exponential decay or a hyperbolic decay.

$$r_t = \sigma_t \varepsilon_t \ , \quad \sigma_t^2 = \beta_0 + \sum_{j=1}^{\infty} \beta_j r_{t-j}^2 \ ,$$

with the following parameterization

$$X_t = r_t^2 \ , \quad \xi_t = \left(\frac{\varepsilon_t^2 - \lambda_1}{\kappa}, 1 \right) \ , \quad a = \begin{pmatrix} \kappa \beta_0 \\ \lambda_1 \beta_0 \end{pmatrix} \ , \quad a_j = \begin{pmatrix} \kappa \beta_j \\ \lambda_1 \beta_j \end{pmatrix} \ ,$$

where the ε are centered and iid, $\lambda_1 = \mathbb{E}(\varepsilon_0^2)$, and $\kappa^2 = \mathrm{var}(\varepsilon_0^2)$. Note that the first coordinate of ξ_0 is thus a centered random variable. Conditions for stationarity of the unidimensional ARCH(∞) model have been derived using Volterra expansions by Giraitis *et al.* [GKL00] and Giraitis and Surgailis [GS02]. The present paper is a multidimensional generalization of these previous works.

(A5). We can consider models with several innovations and variables such as:

$$Z_t = \zeta_{1,t}\left(\alpha + \sum_{j=1}^{\infty}\alpha_j^1 Z_{t-j}\right) + \mu_{1,t}\left(\beta + \sum_{j=1}^{\infty}\beta_j^1 Y_{t-j}\right) + \gamma + \sum_{j=1}^{\infty}\gamma_j^1 Z_{t-j}\ ,$$

$$Y_t = \zeta_{2,t}\left(\alpha + \sum_{j=1}^{\infty}\alpha_j^2 Y_{t-j}\right) + \mu_{2,t}\left(\beta + \sum_{j=1}^{\infty}\beta_j^2 Z_{t-j}\right) + \gamma + \sum_{j=1}^{\infty}\gamma_j^2 Y_{t-j}\ .$$

This model is straightforwardly described through equation (1) with $d = 2$ and $m = 3$. Here $\xi_t = \begin{pmatrix} \zeta_{1,t} & \mu_{1,t} & 1 \\ \zeta_{2,t} & \mu_{2,t} & 1 \end{pmatrix}$ is a 2×3 iid sequence, $a_j = \begin{pmatrix} \alpha_j^1 & \alpha_j^2 \\ \beta_j^1 & \beta_j^2 \\ \gamma_j^1 & \gamma_j^2 \end{pmatrix}$ is a 3×2 matrix and $a = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ is a vector in $\mathbb{R}^3$ and the process $X_t = \begin{pmatrix} Z_t \\ Y_t \end{pmatrix}$ is a vector of dimension 2. Dimensions $m = 3$ and $d = 2$ are only set here for simplicity. Replacing $m = 3$ by $m = 6$ would allow to consider different coefficients α, β and γ for both lines in this system of two coupled equations.

This generalizes the class of multivariate ARCH(∞) processes, defined in the p-dimensional case as:

$$R_t = \Sigma_t^{\frac{1}{2}}\varepsilon_t\ ,$$

where R_t is a p–dimensional vector, Σ_t is a $p \times p$ positive definite matrix, and ε_t is a p–dimensional vector. Those models are formally investigated by Farid Boussama in [Bou98]; published references include [Bou00] and [EK96].

This model is of interest in financial econometrics as the volatility of asset prices of linked markets, e.g., major currencies in the Foreign Exchange (FX) market, are correlated, and in some cases display a common strong dependence structure; see [Tey97]. This common dependence structure can be modeled with the assumption that the innovations $\varepsilon_1, \ldots, \varepsilon_p$ are correlated. An (empirically) interesting case for the bivariate model (X_t, Y_t) is obtained with the assumption that the $(\zeta_{1,t}, \zeta_{2,t})$ are cross-correlated.

2 Existence and Uniqueness in L^p

In the sequel, we set $A(x) = \sum_{j\geq x}\|a_j\|$, $A = A(1)$, where $\|\cdot\|$ denotes the matrix norm.

Theorem 1. *Let $p > 0$, we denote*

$$\varphi = \sum_{j\geq 1} \|a_j\|^{p\wedge 1} \left(\mathbb{E}\|\xi_0\|^p\right)^{\frac{1}{p\wedge 1}} \ . \tag{2}$$

If $\varphi < 1$, then a stationary solution in L^p to equation (1) *is given by:*

$$X_t = \xi_t \left(a + \sum_{k=1}^{\infty} \sum_{j_1,\ldots,j_k\geq 1} a_{j_1}\xi_{t-j_1}\cdots a_{j_k}\xi_{t-j_1-\cdots-j_k}\cdot a \right) \ . \tag{3}$$

Proof. The norm used for the matrices is any multiplicative norm. We have to show that expression (3) is well defined under the conditions stated above, converges absolutely in L^p, and that it satisfies equation (1).
Step 1. We first show that expression (3) is well defined (after the second line we omit to precise the norms). For $p \geq 1$, we have

$$\begin{aligned}
&\sum_{j_1,\ldots,j_k\geq 1} \|a_{j_1}\xi_{t-j_1}\cdots a_{j_k}\xi_{t-j_1-\cdots-j_k}\|_{m\times m} \\
&\qquad \leq \sum_{j_1,\ldots,j_k\geq 1} \|a_{j_1}\|_{m\times d}\cdots\|a_{j_k}\|_{m\times d}\|\xi_{t-j_1}\|_{d\times m}\cdots\|\xi_{t-j_1-\cdots-j_k}\|_{d\times m} \ .
\end{aligned}$$

The series thus converges in norm L^p because

$$\begin{aligned}
&\sum_{k=1}^{\infty} \sum_{j_1,\ldots,j_k\geq 1} \left(\mathbb{E}\|a_{j_1}\xi_{t-j_1}\cdots a_{j_k}\xi_{t-j_1-\cdots-j_k}\|^p\right)^{1/p} \\
&\leq \sum_{k=1}^{\infty} \sum_{j_1,\ldots,j_k\geq 1} \|a_{j_1}\|\cdots\|a_{j_k}\|(\mathbb{E}\|\xi_{t-j_1}\|^p)^{1/p}\cdots(\mathbb{E}\|\xi_{t-j_1-\cdots-j_k}\|^p)^{1/p} \\
&\leq \sum_{k=1}^{\infty} \sum_{j_1,\ldots,j_k\geq 1} \|a_{j_1}\|\cdots\|a_{j_k}\| \left(\mathbb{E}\|\xi_0\|^p\right)^{\frac{k}{p}} \leq \sum_{k=1}^{\infty} \varphi^k \ .
\end{aligned}$$

The series $\sum_{k=1}^{\infty} \varphi^k$ is finite since $\varphi < 1$, hence the series (3) converges in L^p.

For $p < 1$, the convergence is defined through the metric $d_p(U,V) = \mathbb{E}\|U-V\|^p$ between vector valued L^p random variables U, V and we start from

$$\begin{aligned}
&\left(\sum_{j_1,\ldots,j_k\geq 1} \|a_{j_1}\xi_{t-j_1}\cdots a_{j_k}\xi_{t-j_1-\cdots-j_k}\| \right)^p \\
&\qquad\qquad \leq \sum_{j_1,\ldots,j_k\geq 1} \|a_{j_1}\xi_{t-j_1}\cdots a_{j_k}\xi_{t-j_1-\cdots-j_k}\|^p \ ,
\end{aligned}$$

and we use the same arguments as for $p = 1$.
Step 2. We now show that equation (3) is solution to equation (1):

$$X_t = \xi_t \left(a + \sum_{k=1}^{\infty} \sum_{j_1,\ldots j_k\geq 1} a_{j_1}\xi_{t-j_1}\cdots a_{j_k}\xi_{t-j_1-\cdots-j_k}\cdot a \right)$$

$$
\begin{aligned}
&= \xi_t \left(a + \sum_{j\geq 1} a_j \xi_{t-j} + \right. \\
&\quad \left. + \sum_{k=2}^{\infty} \sum_{j_1 \geq 1} a_j \xi_{t-j} \sum_{j_2,\ldots,j_k \geq 1} a_{j_2} \xi_{t-j-j_2} \cdots a_{j_k} \xi_{t-j-j_2-\cdots-j_k} \cdot a \right) \\
&= \xi_t \left(a + \sum_{j\geq 1} a_j \times \right. \\
&\quad \left. \xi_{t-j} \Big\{ a \ + \sum_{k=2}^{\infty} \sum_{j_2,\ldots,j_k \geq 1} a_{j_2} \xi_{(t-j)-j_2} \cdots a_{j_k} \xi_{(t-j)-j_2-\cdots-j_k} \cdot a \Big\} \right) \\
&= \xi_t \Big(a + \sum_{j\geq 1} a_j X_{t-j} \Big) .
\end{aligned}
$$

Remark 1. The uniqueness of this solution is not demonstrated without additional condition; see Theorem 2 and section 5 below.

Theorem 2. *Assume that $p \geq 1$ then from (2), $\varphi = \sum_j \|a_j\| \|\xi_0\|_p$. Assume $\varphi < 1$. If a stationary solution $(Y_t)_{t\in\mathbb{Z}}$ to equation (1) exists (a.s.), if Y_t is independent of the sigma-algebra generated by $\{\xi_s; s > t\}$, for each $t \in \mathbb{Z}$, then this solution is also in L^p and it is (a.s.) equal to the previous solution $(X_t)_{t\in\mathbb{Z}}$ defined by equation (3).*

Proof. Step 1. We first prove that $\|Y_0\|_p < \infty$. From equation (1) and from $\{Y_t\}_{t\in\mathbb{Z}}$'s stationarity, we derive

$$
\|Y_0\|_p \leq \|\xi_0\|_p \left(\|a\| + \sum_{j=1}^{\infty} \|a_j\| \|Y_0\|_p \right) < \infty ,
$$

hence, the first point in the theorem follows from:

$$
\|Y_0\|_p \leq \frac{\|\xi_0\|_p \|a\|}{1-\varphi} < \infty .
$$

Step 2. As in [GKL00] we write recursively $Y_t = \xi_t \left(a + \sum_{j\geq 1} a_j Y_{t-j} \right) = X_t^m + S_t^m$, with

$$
\begin{aligned}
X_t^m &= \xi_t \left(a + \sum_{k=1}^{m} \sum_{j_1,\cdots,j_k \geq 1} a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1 \cdots - j_k} a \right) , \\
S_t^m &= \xi_t \left(\sum_{j_1,\cdots,j_{m+1} \geq 1} a_{j_1} \xi_{t-j_1} \cdots a_{j_m} \xi_{t-j_1 \cdots - j_m} a_{j_m+1} Y_{t-j_1 \cdots - j_m} \right) .
\end{aligned}
$$

We have

$$\|S_t^m\|_p \le \|\xi\|_p \sum_{j_1,\cdots,j_{m+1}\ge 1} \|a_{j_1}\|\cdots\|a_{j_{m+1}}\|\|\xi\|_p^m\|Y_0\|_p = \|Y_0\|_p\varphi^{m+1}\ .$$

We recall the additive decomposition of the chaotic expansion X_t in equation (3) as a finite expansion plus a negligible remainder that can be controlled $X_t = X_t^m + R_t^m$ where

$$R_t^m = \xi_t \left(\sum_{k>m} \sum_{j_1,\cdots,j_k\ge 1} a_{j_1}\xi_{t-j_1}\cdots a_{j_k}\xi_{t-j_1\cdots-j_k}a \right)\ ,$$

satisfies

$$\|R_t^m\|_p \le \|a\|\|\xi_0\|_p \sum_{k>m}\varphi^k \le \|a\|\|\xi_0\|_p \frac{\varphi^m}{1-\varphi} \to 0\ .$$

Then, the difference between those two solutions is controlled as a function of m with $X_t - Y_t = R_t^m - S_t^m$, hence

$$\begin{aligned}\|X_t - Y_t\|_p &\le \|R_t^m\|_p + \|S_t^m\|_p \\ &\le \frac{\varphi^m}{1-\varphi}\|a\|\|\xi_0\|_p + \|Y_0\|_p\varphi^m \le 2\frac{\varphi^m}{1-\varphi}\|a\|\|\xi_0\|_p\ ,\end{aligned}$$

and thus $Y_t = X_t$ a.s.

We also consider the following extension of equation (1) to random fields $\{X_t\}_{t\in\mathbb{Z}^D}$:

Lemma 1. *Assume that a_j are $m\times d$-matrices now defined for each $j\in\mathbb{Z}^D\setminus\{0\}$. Fix an arbitrary norm $\|\cdot\|$ on $\mathbb{Z}^D$. We extend the previous function A to $A(x)=\sum_{\|j\|\ge x}\|a_j\|$, $A=A(1)$ and we suppose with $p=\infty$ that $\varphi = A\|\xi_0\|_\infty < 1$. Then the random field*

$$X_t = \xi_t \left(a + \sum_{k=1}^{\infty}\sum_{j_1\neq 0}\cdots\sum_{j_k\neq 0} a_{j_1}\xi_{t-j_1}\cdots a_{j_k}\xi_{t-j_1-\cdots-j_k}a \right) \tag{4}$$

is a solution to the recursive equation:

$$X_t = \xi_t\left(a + \sum_{j\neq 0} a_j X_{t-j}\right)\ ,\quad t\in\mathbb{Z}^D\ . \tag{5}$$

Moreover, each stationary solution to this equation is also bounded and equals X_t, a.s.

The proof is the same as before, we first prove that any solution is bounded and we expand it as the sum of the first terms in this chaotic expansion, up to a small remainder (wrt to sup norm); the only important modification follows from the fact that now $j_1+\cdots+j_\ell$ may really vanish for nonzero j_i's which entails that the bound with expectation has to be replaced by upper bounds.

Remark 2. In the previous lemma, the independence of the ξ's does not play a role. We may have stated it for arbitrary random fields $\{\xi_t\}$ such that $\|\xi_t\|_\infty \le M$ for each $t \in \mathbb{Z}^D$; such models with dependent inputs are interesting but assumptions on the innovations are indeed very strong. This means that such models are heteroscedastic but with bounded innovations: according to [MH04], this restriction excludes extreme phenomena like crashes and bubbles. Mandelbrot school has shown from the seminal paper [Man63] that asset prices returns do not have a Gaussian distribution as the number of extreme deviations, the so–called "Noah effects", of asset returns is far greater than what is allowed by the Normal distribution, even with ARCH–type effects. It is the reason why this extension is not pursued in the present paper.

3 Approximations

This section is aimed to approximate a sequence $\{X_t\}$ given by (3), solution to eqn. (1) by a sequence $\{\tilde{X}_t\}$. We shall prove that we can control the approximation error $\mathbb{E}\|X_t - \tilde{X}_t\|$ within reasonable small bounds.

3.1 Approximation by Independence

The purpose is to approximate X_t by a random variable independent of X_0. We set

$$\tilde{X}_t = \xi_t \left(a + \sum_{k=1}^{\infty} \sum_{j_1+\cdots+j_k<t} a_{j_1}\xi_{t-j_1} \cdots a_{j_k}\xi_{t-j_1-\cdots-j_k} a \right) .$$

Proposition 1. *Define φ from* (2). *A bound for the error is given by:*

$$\mathbb{E}\|X_t - \tilde{X}_t\| \le \mathbb{E}\|\xi_0\| \left(\mathbb{E}\|\xi_0\| \sum_{k=1}^{t-1} k\varphi^{k-1} A\left(\frac{t}{k}\right) + \frac{\varphi^t}{1-\varphi} \right) \|a\| .$$

Furthermore, we have as particular results that if b, $C > 0$ *and* $q \in [0,1)$, *then for a suitable choice of constants* K, K':

$$\mathbb{E}\|X_t - \tilde{X}_t\| \le \begin{cases} K\frac{(\log(t))^{b\vee 1}}{t^b} , & \textit{for Riemannian decays } A(x) \le Cx^{-b} , \\ K'(q\vee\varphi)^{\sqrt{t}} , & \textit{for geometric decays } A(x) \le Cq^x . \end{cases}$$

Remark 3. Note that in the first case this decay is essentially the same Riemannian one while it is sub-geometric (like $t \mapsto e^{-c\sqrt{t}}$) when the decay of the coefficients is geometric.

Remark 4. In the paper Riemannian or Geometric decays always refer to the previous relations.

Idea of the Proof. A careful study of the terms in X_t's expansion which do not appear in $\tilde{X}_t$ entails the following bound with the triangular inequality. For this, quote that if $j_1 + \cdots + j_k \geq t$ for some $k \geq 1$ then, at least, one of the indices $j_1, \ldots,$ or j_k is larger than t/k. The additional term corresponds to those terms with indices $k > t$ in the expansion (3).

The following extension to the case of the random fields determined in lemma 1 is immediate by setting

$$\tilde{X}_t = \xi_t \left(a + \sum_{k=1}^{\infty} \sum_{\substack{j_1, \ldots, j_k \neq 0 \\ \|j_1\| + \cdots + \|j_k\| < \|t\|}} a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1-\cdots-j_k} a \right) .$$

Proposition 2. *The random field $(X_t)_{t \in \mathbb{Z}^D}$ defined in lemma 1 satisfies:*

$$\mathbb{E}\|X_t - \tilde{X}_t\| \leq \mathbb{E}\|\xi_0\| \left(\|\xi_0\|_\infty \sum_{1 \leq k < \|t\|} k\varphi^{k-1} A\left(\frac{\|t\|}{k}\right) + \frac{\varphi^{\|t\|}}{1-\varphi} \right) \|a\| .$$

3.2 Coupling

First note that the variable $\tilde{X}_t$ which approximates X_t does not follow the same distribution. For dealing with this issue, it is sufficient to construct a sequence of iid random variables ξ'_i which follow the same distribution as the one of the ξ_i, each term of the sequence being independent of all the ξ_i. We then set

$$\xi^*_t = \begin{cases} \xi_t \text{ if } t > 0 , \\ \xi'_t \text{ if } t \leq 0 , \end{cases}$$

$$X^*_t = \xi_t \left(a + \sum_{k=1}^{\infty} \sum_{j_1, \ldots, j_k} a_{j_1} \xi^*_{t-j_1} \cdots a_{j_k} \xi^*_{t-j_1-\cdots-j_k} a \right) .$$

Coefficients τ_t for the τ–dependence introduced by Dedecker and Prieur [DP01] are easily computed. In this case, we find the upper bounds from above, up to a factor 2:

$$\tau_t = \mathbb{E}\|X_t - X^*_t\| \leq 2\mathbb{E}\|\xi_0\| \left(\mathbb{E}\|\xi_0\| \sum_{k=1}^{t-1} k\varphi^{k-1} A\left(\frac{t}{k}\right) + \frac{\varphi^t}{1-\varphi} \right) \|a\| ;$$

see also Rüschendorf [RüS], Prieur [Pri05]. These coefficients τ_k are defined as $\tau_k = \tau(\sigma(X_i, i \leq 0), X_k)$ where for each random variable X and each σ-algebra $\mathcal{M}$ one sets

$$\tau(\mathcal{M}, X) = \mathbb{E}\left\{ \sup_{\mathrm{Lip} f \leq 1} \left| \int f(x) \mathbb{P}_{X|\mathcal{M}}(dx) - \int f(x) \mathbb{P}_X(dx) \right| \right\}$$

where $\mathbb{P}_X$ and $\mathbb{P}_{X|\mathcal{M}}$ denotes the distribution and the conditional distribution of X on the σ–algebra $\mathcal{M}$ and $\mathrm{Lip} f = \sup_{x \neq y} |f(x) - f(y)|/\|x - y\|$.

3.3 Markovian Approximation

We consider equation (1) truncated at the order N: $Y_t = \xi_t(a + \sum_{j=1}^{N} a_j Y_{t-j})$. The solution considered above can be rewritten as

$$X_t^N = \xi_t \left(a + \sum_{k=1}^{\infty} \sum_{N \geq j_1, \ldots, j_k \geq 1} a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1-\cdots-j_k} a \right) .$$

We can easily find an upper bound of the error: $\mathbb{E}\|X_t - X_t^N\| \leq \sum_{k=1}^{\infty} A(N)^k$. As in proposition 1, in the Riemannian case, this bound of the error writes as $C \sum_{k=1}^{\infty} N^{-bk} \leq C/(N^b - 1)$ with $b > 1$, while in the geometric case, this writes as $Cq^N/(1-q^N) \leq Cq^N/(1-q)$, $0 < q < 1$.

4 Weak Dependence

Consider integers $u, v \geq 1$. Let $i_1 < \cdots < i_u$, $j_1 < \cdots < j_v$ be integers with $j_1 - i_u \geq r$, we set U and V for the two random vectors $U = (X_{i_1}, X_{i_2}, \ldots, X_{i_u})$ and $V = (X_{j_1}, X_{j_2}, \ldots, X_{j_v})$. We fix a norm $\|\cdot\|$ on $\mathbb{R}^d$. For a function $h : \left(\mathbb{R}^d\right)^w \to \mathbb{R}$ we set

$$\mathrm{Lip}(h) = \sup_{x_1, y_1, \ldots, x_w, y_w \in \mathbb{R}^d} \frac{|h(x_1, \ldots, x_w) - h(y_1, \ldots, y_w)|}{\sum_{i=1}^{w} \|x_i - y_i\|} .$$

Theorem 3. *Assume that the coefficient defined by* (2) *satisfies* $\varphi < 1$. *The solution* (3) *to the equation* (1) *is* $\theta-$*weakly dependent, see [DD03]. This means that:*

$$|\mathrm{cov}(f(U), g(V))| \leq 2v\|f\|_\infty \mathrm{Lip}(g)\theta_r ,$$

for any integers $u, v \geq 1$, $i_1 < \cdots < i_u$, $j_1 < \cdots < j_v$ *such that* $j_1 - i_u \geq r$*; with*

$$\theta_r = \mathbb{E}\|\xi_0\| \left(\mathbb{E}\|\xi_0\| \sum_{k=1}^{r-1} k\varphi^{k-1} A\left(\frac{r}{k}\right) + \frac{\varphi^r}{1-\varphi} \right) \|a\| .$$

Proof. For calculating a weak dependence bound, we approximate the vector V by the vector $\hat{V} = (\hat{X}_{j_1}, \hat{X}_{j_2}, \ldots, \hat{X}_{j_v})$, where we set

$$\hat{X}_t = \xi_t \left(a + \sum_{k=1}^{\infty} \sum_{j_1 + \cdots + j_k < s} a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1-\cdots-j_k} a \right) .$$

Note that for each index $j \in \mathbb{Z}$, $\hat{X}_j$ is independent of $(X_{j-s})_{s \geq r}$. Note that for $1 \leq k \leq v$, $\mathbb{E}\|X_{j_k} - \hat{X}_{j_k}\| \leq \theta_r$ defined in theorem 3. Then

$$
\begin{aligned}
|\mathrm{cov}(f(U),g(V))| &\leq \left|\mathbb{E}\left(f(U)(g(V)-g(\hat{V})\right)-\mathbb{E}(f(U))\mathbb{E}(g(V)-g(\hat{V}))\right| \\
&\leq 2\|f\|_\infty \mathbb{E}\left|g(V)-g(\hat{V})\right| \\
&\leq 2\|f\|_\infty \mathrm{Lip}(g)\sum_{k=1}^{v}\mathbb{E}\|X_{j_k}-\hat{X}_{j_k}\| \\
&\leq 2v\|f\|_\infty \mathrm{Lip}(g)\theta_r \ .
\end{aligned}
$$

Remark 5. We obtain explicit expressions for this bound in Proposition 1 for the Riemannian and geometric decay rates.

Remark 6. In the case of random fields the η-weak dependence condition in [DL99] or [DL02] holds in a similar way with

$$\eta_r = 2\mathbb{E}\|\xi_0\|\left(\|\xi_0\|_\infty \sum_{k<r/2} k\varphi^{k-1}A\left(\frac{r}{k}\right)+\frac{\varphi^{[r/2]}}{1-\varphi}\right)\|a\| \ ,$$

which means that the previous bound now reads

$$|\mathrm{cov}(f(U),g(V))| \leq \big(u\|g\|_\infty \mathrm{Lip}(f)+v\|f\|_\infty \mathrm{Lip}(g)\big)\eta_r \ .$$

The argument is the same except for the fact that now $\hat{U}$ and $\hat{V}$ are independent vectors with truncations at a level $s=[r/2]$ but $\hat{V}$ and U are not necessarily independent (recall that independence of U and $\hat{V}$ follows from $s \geq r$ in the proof for the causal case). This point makes the previous bound a bit more complicated than the one in theorem 3 and it explains the appearance of the factor 2 in the expression of η_r.

Remark 7. These weak dependence conditions imply various limit theorems both for partial sums processes and for the empirical process (see [DL99], [DD03] and [DL02]).

5 L^2 Properties

For the univariate case, the uniqueness of a stationary solution to (1) has been proved by [GKL00]. We present a criterion for existence and uniqueness of a solution in L^2. This solution is no longer necessarily weakly dependent.

Theorem 4. *Assume that the iid sequence $\{\xi_t\}$ is centered and the spectral radius $\rho(S)$ of the matrix $S=\sum_{k=1}^{\infty} a_k'\mathbb{E}(\xi_k'\xi_k)a_k$ satisfies $\rho(S)<1$. Then there exists a unique stationary solution in L^2 to equation* (1) *given by* (3).

Remark 8.

- The assumption $\rho(S)<1$ implies $\xi_t \in L^2$ for $t \in \mathbb{Z}$.

- The bilinear model of Example 2 is shown in [GS02] to display the double long memory property when the series $\{\alpha_j\}$ and $\{\beta_j\}$ are not summable but satisfy the condition

$$\sum_{j=1}^{\infty}\left(\alpha_j^2\mathbb{E}\zeta_0^2+\beta_j^2\right)<1\ .$$

  As a particular case, the squares of the LARCH(∞) process in Example 1 display long–range dependence. [GS02] prove that the corresponding partial sums process converges to the fractional Brownian Motion with normalization $\gg\sqrt{n}$.
- The GARCH(p,q) models in example 3, are always weakly dependent, in the sense of [DL99].
- Note that [GKL00] and [GS02] prove that the stationary ARCH(∞) model (Example 4), is not long range dependent in the previous sense; more precisely the partial sums process, normalized with $\sqrt{n}$, converges to the Brownian Motion.

Proof. Step 1: existence. Define $T=\mathbb{E}(\xi_k'\xi_k)$. Consider the chaotic solution (3) and set

$$C_t(k_2,\ldots,k_\ell)=\xi_t a_{k_2}\xi_{t-k_2}\cdots a_{k_\ell}\xi_{t-k_2\cdots-k_\ell}a\ .$$

Write $\mathbb{E}(X_t'X_t)=a'\mathbb{E}\xi_t'\xi_t a+B=a'Ta+B$, where

$$\begin{aligned}
B&=\sum_{\ell,k_1,\ldots,k_\ell\ge1}\mathbb{E}C_{t-k_1}'(k_2,\ldots,k_\ell)a_{k_1}'Ta_{k_1}C_{t-k_1}(k_2,\ldots,k_\ell)\\
&=\sum_{\ell,k_1,\ldots,k_\ell}\mathbb{E}C_{t-k_1}'(k_2,\ldots,k_\ell)a_{k_1}'\mathbb{E}\xi_{t-k_1}'\xi_{t-k_1}a_{k_1}C_{t-k_1}(k_2,\ldots,k_\ell)\\
&=\sum_{\ell,k_1,\ldots,k_\ell}\mathbb{E}C_{t-k_1}'(k_2,\ldots,k_\ell)\left(\mathbb{E}a_{k_1}'\xi_{t-k_1}'\xi_{t-k_1}a_{k_1}\right)C_{t-k_1}(k_2,\ldots,k_\ell)\\
&=\sum_{\ell,k_1,\ldots,k_\ell}\mathbb{E}C_t'(k_2,\ldots,k_\ell)\left(\mathbb{E}a_{k_1}'\xi_{t-k_1}'\xi_{t-k_1}a_{k_1}\right)C_t(k_2,\ldots,k_\ell)\\
&=\sum_{\ell,k_2,\ldots,k_\ell}\mathbb{E}C_t'(k_2,\ldots,k_\ell)SC_t(k_2,\ldots,k_\ell)\\
&\le\rho(S)\sum_{\ell,k_2,\ldots,k_\ell}\mathbb{E}C_t'(k_2,\ldots,k_\ell)C_t(k_2,\ldots,k_\ell)\\
&\le\mathbb{E}(\xi_0a)'(\xi_0a)\sum_{\ell=1}^{\infty}\rho(S)^\ell\qquad\text{(recursively)}\\
&\le a'a\rho(T)\sum_{\ell=1}^{\infty}\rho(S)^\ell\ ,
\end{aligned}$$

hence,

$$\mathbb{E}(X_t'X_t) \le a'Ta + a'a\frac{\rho(T)}{1-\rho(S)} < \infty \ . \tag{6}$$

In the previous relations we used both the fact that the ξ_t are centered and iid and the relation $v'Av \le v'v\rho(A)$ which holds if A denotes a non-negative $d \times d$ matrix and $v \in \mathbb{R}^d$. This conclude the proof of the existence of a solution in L^2.

Step 2: L^2 uniqueness. Let now X_t^1 and X_t^2 be two solutions to equation (1) in L^2. Define $\tilde{X}_t = X_t^1 - X_t^2$, then $\tilde{X}_t$ is solution to

$$\tilde{X}_t = \xi_t \tilde{A}_t \ , \quad \tilde{A}_t = \sum_{k=1}^{\infty} a_k \tilde{X}_{t-k} \ . \tag{7}$$

Now we use (7) and the fact that $\tilde{X}_t$ is centered and thus $\mathbb{E}\tilde{X}_s\tilde{X}_t = 0$ for $s \neq t$ to derive

$$\begin{aligned}
\mathbb{E}\left((\tilde{X}_t g)'(\tilde{X}_t g)\right) &= \sum_{k=1}^{\infty} g'\mathbb{E}\left(\tilde{X}_{t-k}' a_{t-k}' T a_{t-k} \tilde{X}_{t-k}\right) g \\
&= \sum_{k=1}^{\infty} g'\mathbb{E}\left(\tilde{X}_t' a_{t-k}' T a_{t-k} \tilde{X}_t\right) g = g'\mathbb{E}\left(\tilde{X}_t' S \tilde{X}_t\right) g \\
&= \mathbb{E}\left((\tilde{X}_t g)' S (\tilde{X}_t g)\right) \le \rho(S)\mathbb{E}\left((\tilde{X}_t g)'(\tilde{X}_t g)\right) \ .
\end{aligned}$$

From equation (6), this expression is finite and thus the assumption $\rho(S) < 1$ concludes the proof.

Remark 9. The proof does not extend to the case of random fields because in this case the previous arguments of independence cannot be used. In that case we cannot address the question of uniqueness.

The previous L^2 existence and uniqueness assumptions do not imply that $\sum_{j\ge1} \|a_j\| < \infty$, thus this situation is perhaps not a weakly dependent one. Giraitis and Surgailis [GS02], prove results both for the partial sums processes of X_t and $X_t^2 - \mathbb{E}X_t^2$. In our vector case the second problem is difficult and will be addressed in a forthcoming work. However X_t is now the increment of a (vector valued-)martingale and thus we partially extend Theorem 6.2 in [GS02], providing a version of Donsker's theorem for partial sums processes of $\{X_t\}$.

Proposition 3. *Let the assumptions of Theorem 4 hold. Define $S_n(t) = \sum_{1\le i\le nt} X_i$ for $0 \le t \le 1$. Then $S_n(t)/\sqrt{\mathrm{var} S_n(t)}$ converges to $\Sigma W(t)$, for $0 \le t \le 1$ and where $W(t)$ is a $\mathbb{R}^d$ valued Brownian motion and Σ is a symmetric non negative matrix such that Σ^2 is the covariance matrix of X_0. The convergence holds for finite dimensional distributions.*

Remark 10.

- The convergence only holds for any k-tuples $(t_1,\ldots,t_k)\in[0,1]^k$ and since the section is related to L^2 properties we cannot use the tightness arguments in [GS02] to obtain the Donsker theorem; indeed tightness is obtained through moment inequalities of order $p>2$. L^p existence conditions are obtained in [GS02] for the bilinear case if $p=4$; the method is based on the diagram formula and does not extend simply to this vector valued case. A bound for the moments of order $p>2$ of the partial sum process $S_n(t)$ can be obtained using Rosenthal inequality, Theorem 2.11 in [HH80], if $\mathbb{E}\|X_t\|^p<\infty$. This inequality would imply the functional convergence in the Skohorod space $D[0,1]$ if $p>2$.
- If $\mathbb{E}\xi_0\neq 0$ (as for the case of the bilinear model in [GS02]), we may write $X_t=\Delta M_t+\mathbb{E}\xi_0\left(a+\sum_{j=1}^\infty a_jX_{t-j}\right)$ where
$$\Delta M_t=(\xi_t-\mathbb{E}\xi_t)\left(a+\sum_{j=1}^\infty a_jX_{t-j}\right)$$
is a martingale increment. This martingale also obeys a central limit theorem. then,
$$n^{-1/2}S_n(t)\to\bar{\Sigma}W(t)\ ,$$
where $W(t)$ is a vector Brownian motion, where $\bar{\Sigma}'\bar{\Sigma}=\Sigma$. If $\mathbb{E}\xi_0=0$ this is a way to prove proposition 3, which is a multi-dimensional extension of the proof in [GS02].
For the case of the bilinear model, Giraitis and Surgailis also prove the (functional) convergence of the previous sequence of process to a Fractional Brownian Motion in [GS02]. For this, Riemannian decays of the coefficients are assumed. The covariance function of the process is also completely determined to prove such results; this is a quite difficult point to extend to our vector valued frame.
- A final comment concerns the analogue for powers of X_t which, if suitably normalized, are proved to converge to some higher order Rosenblatt process in [GS02] for the bilinear case. We have a structural difficulty to extend it; the only case which may reasonably be addressed is the real valued one ($d=1$), but it also presents very heavy combinatorial difficulties. Computations for the covariances of the processes $(X_t^k)_{t\in\mathbb{Z}}$ will be addressed in a forthcoming work in order to extend those results.

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