Change Point in GARCH Models: Asymptotic and Bootstrap Tests

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Organization of the presentation

- Change-point in the GARCH process:
  - Change in the parameters of the process,
  - Change in the distribution of the innovations.
- Tests based on the empirical process of squared residuals,
- Test based on the Generalized likelihood ratio principle,
- Competing tests,
- Application to series of financial returns.

Framework: GARCH(p,q) processes

$$\begin{aligned} R_t &= \sigma_t \varepsilon_t, \quad E(\varepsilon_t) = 0, \quad \mathrm{Var}(\varepsilon_t) = 1, \\ \sigma_t^2 &= \omega + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 + \sum_{j=1}^{q} \alpha_j \varepsilon_{t-j}^2, \end{aligned}$$

Non-homogeneous GARCH(p, q) process:

Change in the parameters at time t_0

$$\begin{aligned} \sigma_t^2 &= \omega + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 + \sum_{j=1}^{q} \alpha_j \varepsilon_{t-j}^2, \quad t = 1, \ldots, t_0, \\ \sigma_t^2 &= \omega^* + \sum_{j=1}^{p} \beta_j^* \sigma_{t-j}^2 + \sum_{j=1}^{q} \alpha_j^* \varepsilon_{t-j}^2, \quad t = t_0 + 1, \ldots, T \\ \omega &\neq \omega^* \text{ or } \alpha_j \neq \alpha_j^* \text{ or } \beta_j \neq \beta_j^*, \quad \text{for some } j. \end{aligned}$$

Non-homogeneous GARCH(p,q) process:

Change in the distribution of the innovations at time t_0

$$\begin{aligned} R_t &= \sigma_t \varepsilon_t, \\ \sigma_t^2 &= \omega + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 + \sum_{j=1}^{q} \alpha_j \varepsilon_{t-j}^2, \\ \varepsilon_t &\sim D_1(0,1) \quad t = 1, \ldots, t_0, \\ \varepsilon_t &\sim D_2(0,1) \quad t = t_0 + 1, \ldots, T. \end{aligned}$$

Remark: We consider here the case of a single change–point.

Case studied

We consider a GARCH(1,1) process, i.e.,

$$\begin{aligned} R_t &= \sigma_t \varepsilon_t, \quad E(\varepsilon_t) = 0, \quad \mathrm{Var}(\varepsilon_t) = 1, \\ \sigma_t^2 &= \omega + \beta \sigma_{t-1}^2 + \alpha \varepsilon_{t-1}^2. \end{aligned}$$

We are interested in the change in the unconditional variance of the process, i.e., $\omega/(1-\beta-\alpha)$, since:

- the magnitude of the change in this unconditional variance, implied by the changes of the parameters α, β, ω, affects the power of the tests considered here,
- the degree of long range dependence in asset price volatilities might be the consequence of a change in the unconditional variance of the volatility process; the intensity of strong dependence positively depends on the magnitude of this change (see the works by Mikosch and Stărică).

Tests based on the sequential empirical process of squared residuals

Sequential empirical process:

$$\hat{K}_T(s,t) = T^{-1/2} \sum_{1 \le i \le [Ts]} \left[\mathbf{I}\left\{ \hat{\varepsilon}_i^2 \le t \right\} - F(t) \right], \quad 0 < s \le 1.$$

As s increases from 0 to 1, this process “looks” at all potential change–points.

Proposed tests are based on the process:

$$\hat{\mathcal{K}}_T(s,t) = \sqrt{T} \frac{[Ts]}{T} \left(1 - \frac{[Ts]}{T} \right) \left(\hat{F}_{[Ts]}(t) - \hat{F}^*_{T-[Ts]}(t) \right),$$

$$\text{where} \quad \hat{F}_{[Ts]}(t) = \frac{1}{[Ts]} \sum_{1 \le i \le [Ts]} \mathbf{I}\left\{ \hat{\varepsilon}_i^2 \le t \right\}$$

$F^*_{T-[Ts]}(t)$ is defined analogously using the residuals with indexes larger than $[Ts]$.

Tests based on the sequential empirical process of squared residuals (cont.)

Define

$$\hat{\mathcal{K}}(k,t) = \sqrt{T}\,\frac{k}{T}\left(1-\frac{k}{T}\right)\left|\hat{F}_k(t) - \hat{F}_k^*(t)\right|,$$

$$\text{where} \quad \begin{aligned} \hat{F}_k(t) &= \frac{1}{k}\#\{i \le k : \hat{\varepsilon}_i^2 \le t\}, \\ \hat{F}_k^*(t) &= \frac{1}{T-k}\#\{i > k : \hat{\varepsilon}_i^2 \le t\}. \end{aligned}$$

Cramér–von Mises type statistic

From the results in Horváth *et al.* (2001), for large T the Cramér – von Mises statistic defined as

$$\hat{B} := \int_0^1 \left\{ \frac{1}{T} \sum_{i=1}^{T} [\hat{\mathcal{K}}([Ts], \hat{\varepsilon}_i^2)]^2 \right\} ds$$

has approximately the following asymptotic distribution:

$$\hat{B} \sim B := \int_0^1 \int_0^1 \mathcal{K}^2(s,u) du ds,$$

where $\mathcal{K}$ is the "tied-down" Kiefer process. The critical values for B can be derived from Blum, Kiefer and Rosenblatt (1961).

α	$q(\alpha)$	α	$q(\alpha)$
0.1	0.04694	0.005	0.09960
0.05	0.05839	0.001	0.12976
0.01	0.08685	0.0005	0.14290

Table 1: Upper quantiles of B; $P(B > q(\alpha)) = \alpha$.

Kolmogorov-Smirnov type statistic

$$\hat{M}_T = \sup_{0 \leq t \leq \infty} \max_{1 \leq k \leq T} |\hat{\mathcal{K}}_T(k,t)| = \max_{1 \leq k \leq T} \max_{1 \leq j \leq T} | \hat{\mathcal{K}}_T(k, \hat{\varepsilon}_j^2) |$$

Asymptotic distribution of $\hat{M}_T$ is the same as for Picard's (1985) generalized Kolmogorov-Smirnov statistic.

α	c	α	c
0.10	0.715	0.02	0.823
0.05	0.772	0.01	0.874

Table 2: Asymptotic critical values for the Kolmogorov-Smirnov statistic M_T $P(\hat{M}_T > c) \sim \alpha$. Based on Picard (1985).

Generalized Likelihood Ratio tests

Under the null hypothesis, the Data Generating Process (DGP) is:

$$R_t = \sigma_t \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_t^2), \quad \sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha \varepsilon_{t-1}^2,$$

Alternative hypothesis: for $t > t_0$, the DGP is

$$R_t = \sigma_t \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_t^2), \quad \sigma_t^2 = \omega^* + \beta^* \sigma_{t-1}^2 + \alpha^* \varepsilon_{t-1}^2,$$

where $\omega^* \neq \omega$, or $\beta^* \neq \beta$, or $\alpha^* \neq \alpha$.

The GLR statistic is defined by:

$$\Lambda_{t_0} = \frac{\text{maximum likelihood under null}}{\text{maximum likelihood if change at } t_0}.$$

GLR statistics (cont. 1)

Notations

- $\hat{\omega}$, $\hat{\alpha}$, $\hat{\beta}$: the estimates based on $R_1, \ldots, R_T$,
- $\tilde{\omega}$, $\tilde{\alpha}$, $\tilde{\beta}$: the estimates based on $R_1, \ldots, R_{t_0}$,
- $\bar{\omega}$, $\bar{\alpha}$, $\bar{\beta}$: the estimates based on $R_{t_0+1}, \ldots, R_T$.
- Define $\hat{\sigma}_t^2 = \hat{\omega} + \hat{\beta}\hat{\sigma}_{t-1}^2 + \hat{\alpha}\hat{\varepsilon}_{t-1}^2$,
- Define $(\bar{\sigma}_t^2, \tilde{\sigma}_t^2)$ analogously to $\hat{\sigma}_t^2$.

Since we are considering a Gaussian likelihood function, we have:

$$-2\ln\Lambda_{t_0} = -\left[\sum_{t=1}^{t_0}(\ln\tilde{\sigma}_t^2 - \ln\hat{\sigma}_t^2) + \sum_{t=t_0+1}^{T}(\ln\bar{\sigma}_t^2 - \ln\hat{\sigma}_t^2)\right] + \sum_{t=1}^{T}\frac{\hat{\varepsilon}_t^2}{\hat{\sigma}_t^2} - \sum_{t=1}^{t_0}\frac{\hat{\varepsilon}_t^2}{\tilde{\sigma}_t^2} - \sum_{t=t_0+1}^{T}\frac{\hat{\varepsilon}_t^2}{\bar{\sigma}_t^2}$$

GLR statistics (cont. 2)

Since the change–point time t_0 is unknown, we consider:

$$\Lambda_T^* = \max_{1 \leq k \leq T} -2 \ln \Lambda_k$$

Even if the observations are iid, the statistic Λ_T^* exhibits a very interesting behavior:

- it does not converge to a limiting distribution for any normalizing sequence. Instead it satisfies an Erdös type limit theorem with an exponential type extreme value distribution as the limit,
- From Theorem 1.3.1 of Csörgő and Horváth (1997) that for iid observations with estimated d parameters, the asymptotic size α test rejects the null hypothesis of no change in parameters if $\Lambda_T^* > c_n(\alpha)$ where

$$c_n(\alpha) = \frac{[D_d(\log n) - \log[-\log(1-\alpha)] + \log 2]^2}{2 \log \log n},$$

$$\text{with} \quad D_d(x) = 2 \log x + \frac{d}{2} \log \log x - \log \Gamma\left(\frac{d}{2}\right).$$

Remark: The rate of convergence to this limit is rather slow and the asymptotic critical values typically overestimate the finite sample critical values obtained through simulation; see Gombay and Horváth (1996).

GLR statistics (cont. 3)

Weighted likelihood ratio statistic based on the process

$$\left\{\tau(1-\tau)(-2\ln\Lambda_{[T\tau]}),\ 0<\tau<1\right\}$$

If observations are independent with a density depending on b parameters, then the above process can be approximated by $\left\{\sum_{i=1}^{b} B_i^2(\tau),\ 0<\tau<1\right\}$, where $B_i(\cdot), i=1,\ldots,b$ are independent Brownian bridges on the unit interval $[0,1]$.

Under the null hypothesis of constant parameters:

$$\Delta_T^* := T^{-3}\sum_{k=1}^{T-1} k(T-k)(-2\ln\Lambda_k) \xrightarrow{d} \int_0^1 \sum_{i=1}^{b} B_i^2(\tau)d\tau.$$

The critical values for Δ_T^* are derived from Kiefer (1959)

α	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
0.1	0.34730	0.60704	0.84116	1.06311	1.27748
0.05	0.46136	0.74752	1.00018	1.23730	1.46466
0.01	0.74346	1.07366	1.35861	1.62263	1.87215

Table 3: Upper quantiles c defined by $P(\int_0^1 \sum_{i=1}^{b} B_i^2(t)dt > c) = \alpha$.

Bootstrap based inference:

Since we do not known whether the previous results hold for process with weak dependence like GARCH processes, we resort to bootstrap based inference.

Choice for the optimal number of bootstrap by data driven procedure.

Let $\hat{\tau}$ be the statistic of interest:

1. Compute $\hat{\tau}$, set $B = B' = 99$, and the statistics τ_j^*, $j = 1, \ldots, B$.
2. Compute the estimated bootstrap P-value $\hat{P}^*(\hat{\tau})$ based on the B bootstrap samples:

   $$\hat{P}^*(\hat{\tau}) = \frac{1}{B} \sum_{i=1}^{B} \mathbf{I}[\mathcal{T}_{T_i}^* > \hat{\tau}].$$

   Depending whether $\hat{P}^*(\hat{\tau}) < \alpha$ or $\hat{P}^*(\hat{\tau}) > \alpha$, test, at the level $\beta = 0.001$, either the hypothesis that $P^*(\hat{\tau}) \geq \alpha$ or $P^*(\hat{\tau}) \leq \alpha$. If the hypothesis is rejected, then stop, else go to step 3.
3. Set $B = 2B' + 1$. If B is too large, e.g., $B > B_{max} = 12,799$ then stop, else calculate τ_j^* for a further $B' + 1$ bootstrap samples, set $B' = B$ and return to step 2.

Competing tests:

The tests considered here are "posteriori" tests, i.e., are applied when the whole series $R_1, \ldots, R_T$ is observed.

Alternative class of tests: "online" tests:

- We observe the series $R_1, \ldots, R_m$, with $m << T$. On this interval, the process is assumed homogeneous, i.e., its parameters are constant.
- We test for the occurence of a recent change at time $t > m$.

See Mikosch and Stărică (1999), Berkes, Gombay, Horváth and Kokoszka (2002).

We also considered test for a single change-point.

These tests can be adapted to situations of multiple change-points by using the binary segmentation procedure:

- A: Test for the presence of a single change point,
- B: Split the series in two at the detected change-point,
- C: Repeat procedure from A on the two segments, until no further change point is found.

Monte Carlo experiment: Size of the tests

We consider three DGP which satisfy the null hypothesis:

DGP 0a ARCH(1):

$$\omega = 0.1, \quad \beta = 0, \quad \alpha = 0.5.$$

DGP 0b GARCH(1,1):

$$\omega = 0.1, \quad \beta = 0.3, \quad \alpha = 0.3.$$

DGP 0c ARCH(1)

$$\omega = 0.1, \quad \beta = 0, \quad \alpha = 0.8.$$

For **DGP 0c**, the fourth moment does not exist,

Monte Carlo experiment: Power of the tests.
A) changes in parameters

DGP 1 GARCH (1,1) process with change–point in the middle of the sample (large changes in parameters, large change in unconditional variance):

$$\omega = 0.1, \quad \beta = 0.3, \quad \alpha = 0.3, \quad t = 1, \dots, [T/2]$$
$$(\sigma^2 = 0.25),$$
$$\omega = 0.15, \quad \beta = 0.65, \quad \alpha = 0.25, \quad t = [T/2] + 1, \dots, T$$
$$(\sigma^2 = 1.5).$$

DGP 2 GARCH(1,1) process with change–point in the middle of the sample (large changes in parameters, small change in unconditional variance):

$$\omega = 0.1, \quad \beta = 0.3, \quad \alpha = 0.3, \quad t = 1, \dots, [T/2]$$
$$(\sigma^2 = 0.25),$$
$$\omega = 0.125, \quad \beta = 0.6, \quad \alpha = 0.1, \quad t = [T/2] + 1, \dots, T$$
$$(\sigma^2 = 0.4667).$$

DGP 3 GARCH(1,1) process with change–point, such that the unconditional variance remains unchanged ($\sigma^2 = 0.25$)

$$\omega = 0.1, \quad \beta = 0.3, \quad \alpha = 0.3, \quad t = 1, \dots, [T/2],$$
$$\omega = 0.15, \quad \beta = 0.25, \quad \alpha = 0.15, \quad t = [T/2] + 1, \dots, T$$

Monte Carlo experiment: Power of the tests.
A) changes in parameters (cont.)

DGP 4 Smooth transition GARCH(1,1) process,

$$\begin{aligned}\sigma_t^2 &= \omega + \omega^* F(t,[T/2]) + (\beta + \beta^* F(t,[T/2]))\sigma_{t-1}^2 \\ &\quad + (\alpha + \alpha^* F(t,[T/2]))y_{t-1}^2,\end{aligned}$$

with

$$\begin{aligned}\omega &= 0.1, \quad \beta = 0.3, \quad \alpha = 0.3, \\ \omega^* &= 0.05, \quad \beta^* = 0.35, \quad \alpha^* = -0.05, \quad \gamma = 0.05,\end{aligned}$$

- $F(t,k) = (1 + \exp(-\gamma(t-k)))^{-1}$,
- γ strictly positive parameter controlling the smoothness of the transition.

If γ is large, this DGP reduces to **DGP 1**.

With $\gamma = 0.05$, i.e., the transition between the two processes is smooth.

This DGP is of interest for economic variables, the transition of which is smooth, see e.g., Hagerud (1997), Gonzalez-Rivera (1998) and Teräsvirta (1998).

Monte Carlo experiment: Power of the test.
B) change-point near one extreme of the sample.

DGP 5 GARCH(1,1): similar to **DGP 1**, except that the change–point occurs in the first third of the sample:

$$\begin{aligned}\omega &= 0.1, \quad \beta = 0.3, \quad \alpha = 0.3, \quad t = 1, \ldots, [T/3] \\ \omega &= 0.15, \quad \beta = 0.65, \quad \alpha = 0.25 \quad t = [T/3]+1, \ldots, T\end{aligned}$$

DGP 6 GARCH(1,1): similar to **DGP 1**, except that the change–point occurs in the first sixth of the sample:

$$\begin{aligned}\omega &= 0.1, \quad \beta = 0.3, \quad \alpha = 0.3 \quad t = 1, \ldots, [T/6] \\ \omega &= 0.15, \quad \beta = 0.65, \quad \alpha = 0.25 \quad t = [T/6]+1, \ldots, T\end{aligned}$$

Monte Carlo experiment: Power of the test.

C) changes in the innovations

For all cases: GARCH(1,1) with constant parameters but with a change in the distribution of the innovations in the middle of the sample: the parameters are the same as in **DGP 0b**, but the distribution of the innovations ε_t changes as follows:

DGP 7

$$\begin{aligned} \varepsilon_t &\sim N(0,1), \quad t = 1, \ldots, [T/2], \\ \varepsilon_t &\sim t(7), \quad t = [T/2] + 1, \ldots, T \end{aligned}$$

DGP 8 the innovations ε_t for the second half of the sample follow a centralized and standardized $\chi^2(5)$ distribution:

$$\begin{aligned} \varepsilon_t &\sim N(0,1), \quad t = 1, \ldots, [T/2], \\ \varepsilon_t &\sim \chi^2(5), \quad t = [T/2] + 1, \ldots, T \end{aligned}$$

DGP 9 the innovations ε_t for the second half of the sample follow a Laplace (two–sided exponential) distribution:

$$\begin{aligned} \varepsilon_t &\sim N(0,1), \quad t = 1, \ldots, [T/2], \\ \varepsilon_t &\sim \mathrm{Lap}(2^{-1/2}), \quad t = [T/2] + 1, \ldots, T \end{aligned}$$

Conclusions from Monte Carlo results

- GLR tests appear more powerful than tests based on the empirical process of residuals, i.e., CVM and KS tests, except for the case of the change in distribution from the Normal to the Laplacian, which is strongly leptokurtic: the residuals have then a different distribution after and before the change,
- Power of the test increases with the magnitude of the changes in the unconditional variance of the process,
- Asymptotic test Λ has incorrect size and its power is smaller than the power of the bootstrap test Λ,
- the CVM and the Weighted Likelihood Ratio Δ asymptotic tests have the correct size,
- The WLR test Δ is recommended for single change point analysis.

Size-Power curves (Wilk and Gnanadesikan, 1968)

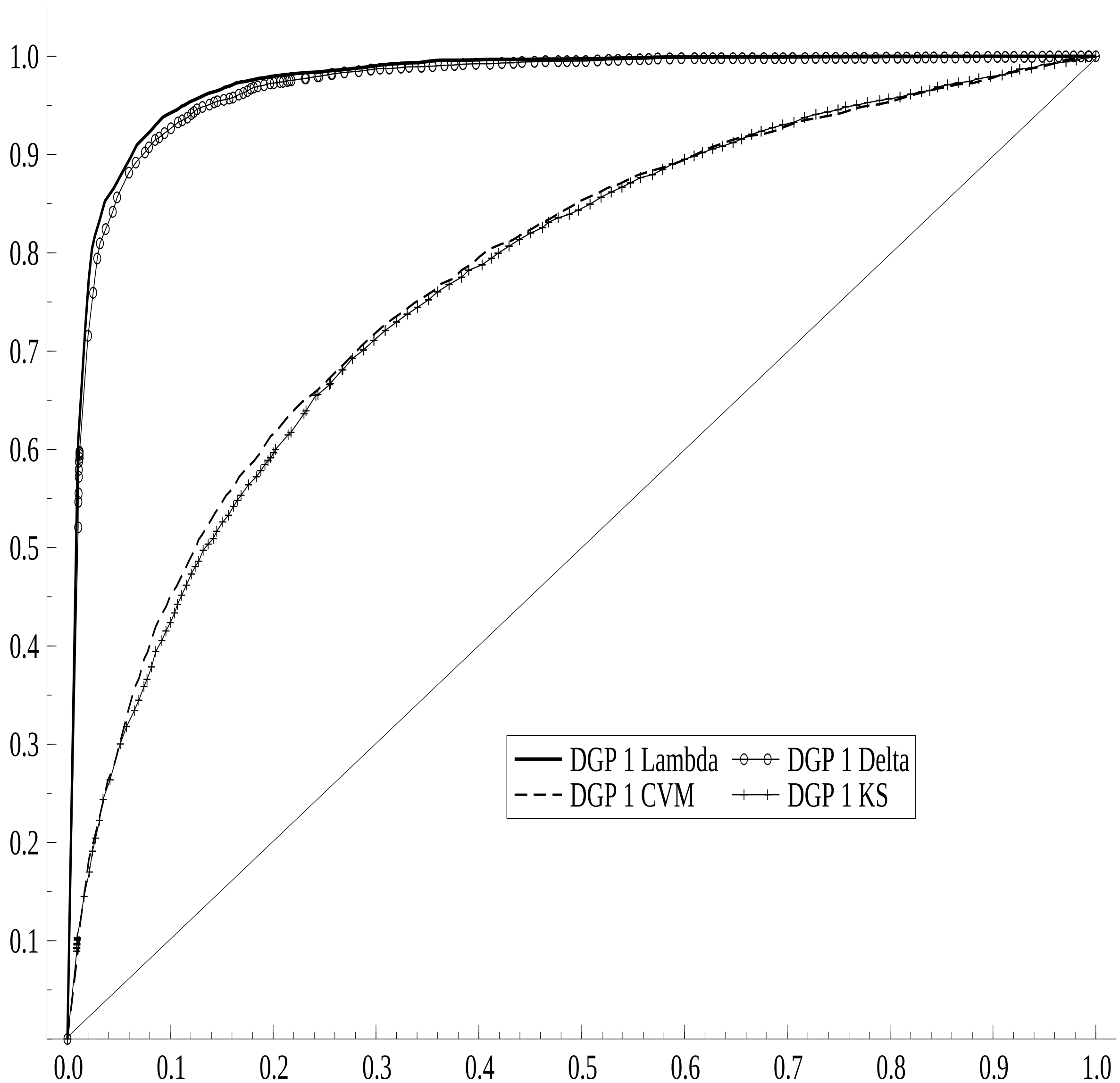

Figure 1: Size–Power Curves. DGP 1.300 Observations

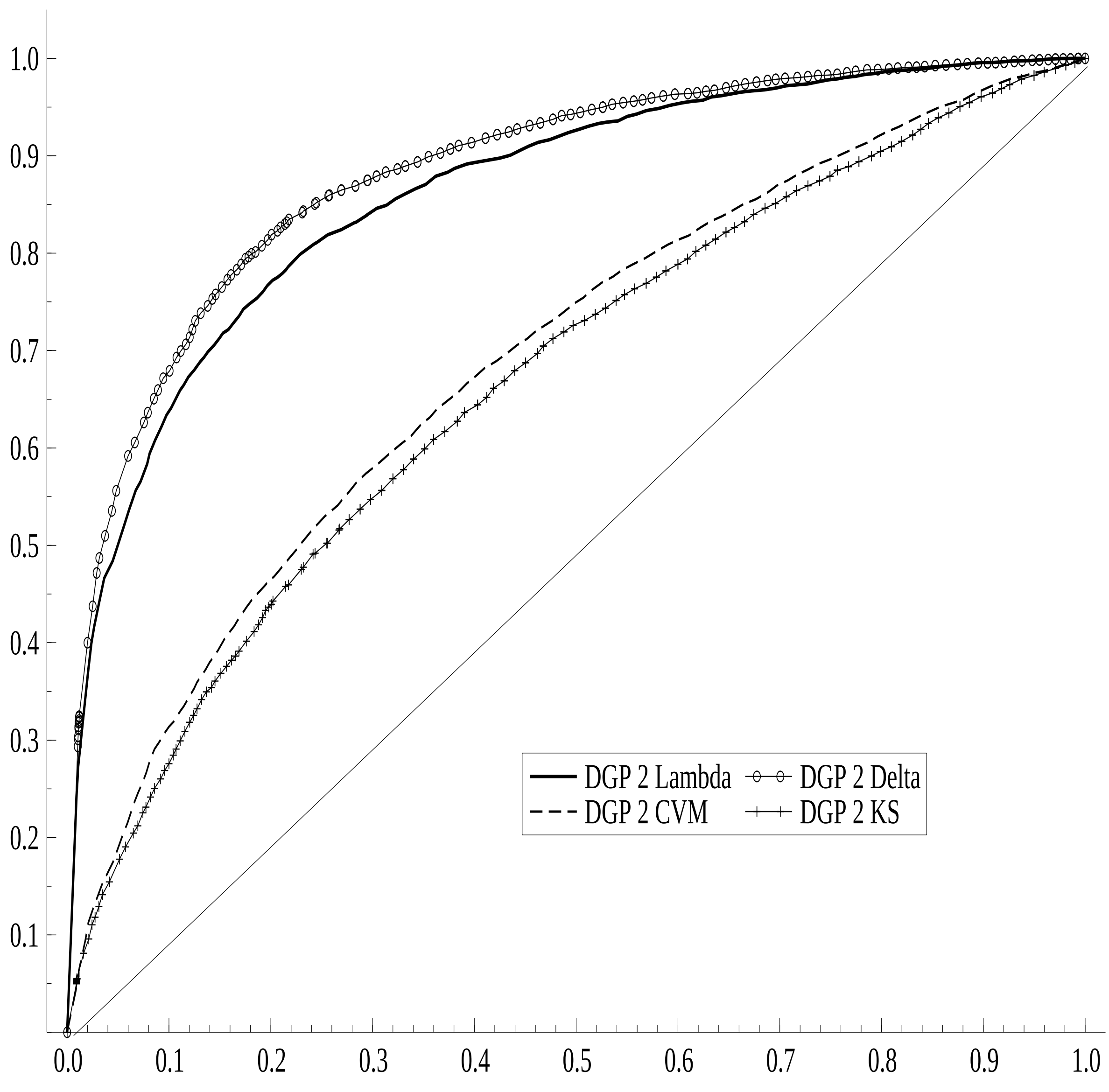

Figure 2: Size–Power Curves. DGP 2.300 Observations

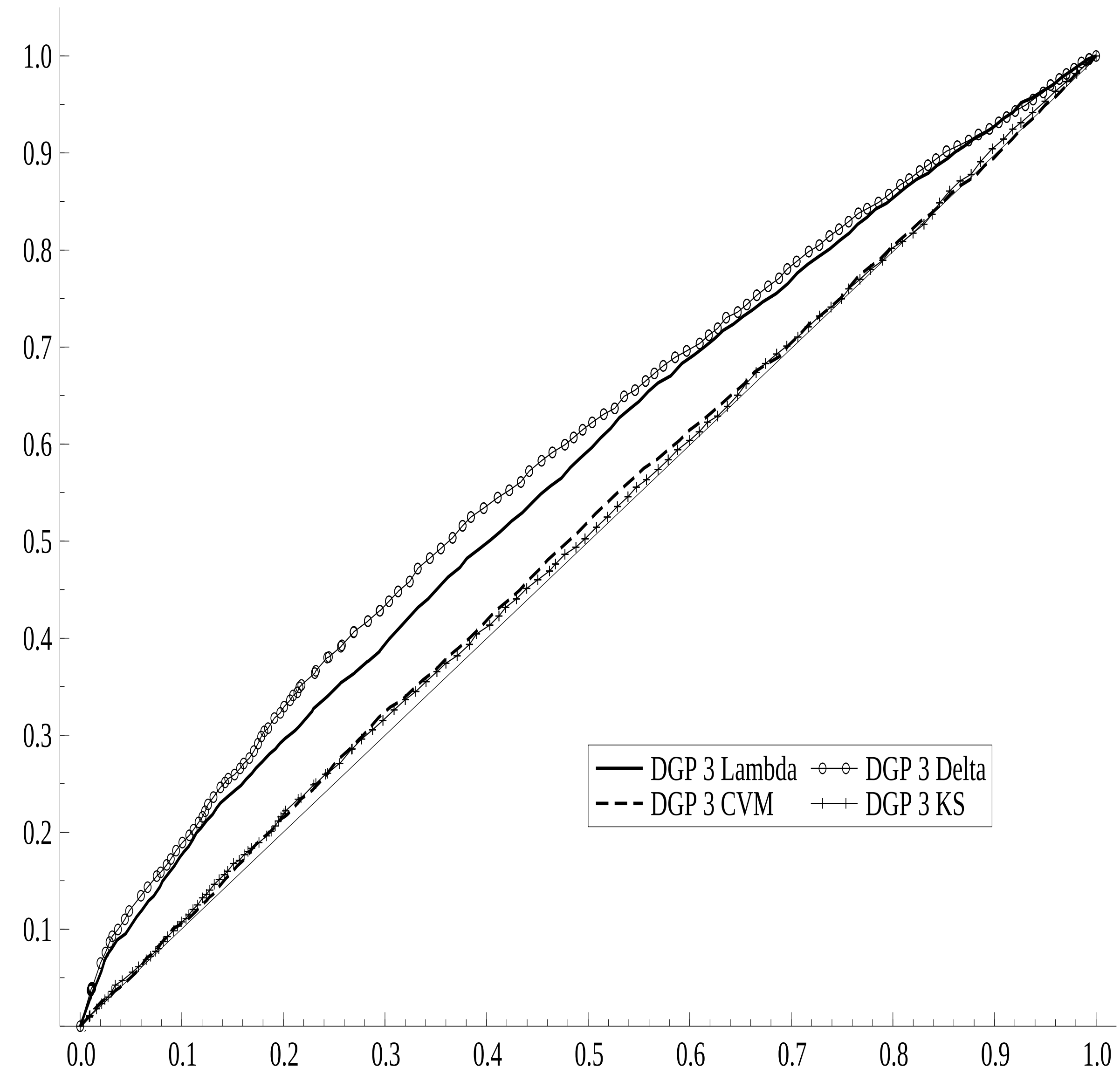

Figure 3: Size–Power Curves. DGP 3.300 Observations

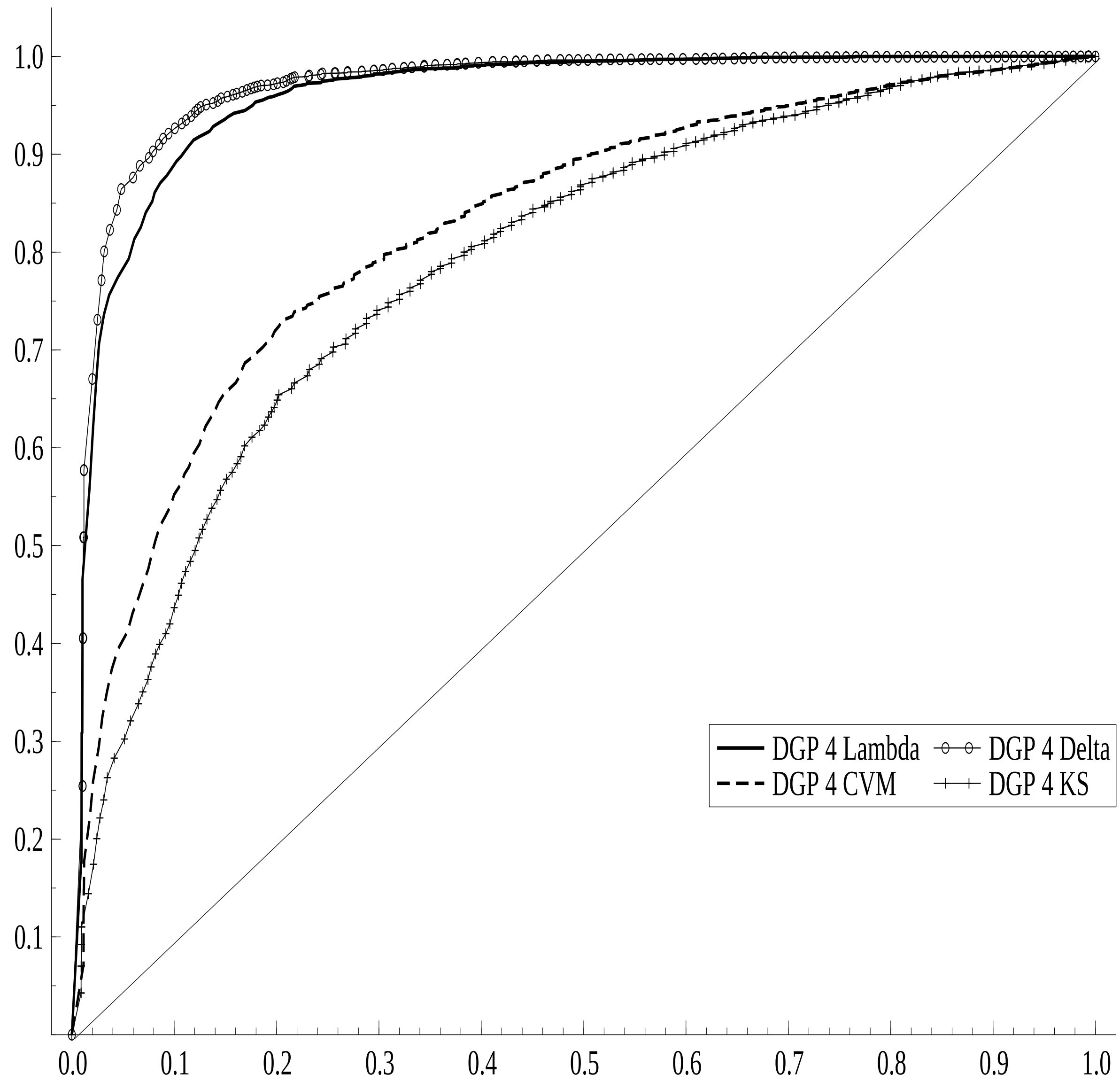

Figure 4: Size–Power Curves. DGP 4.300 Observations

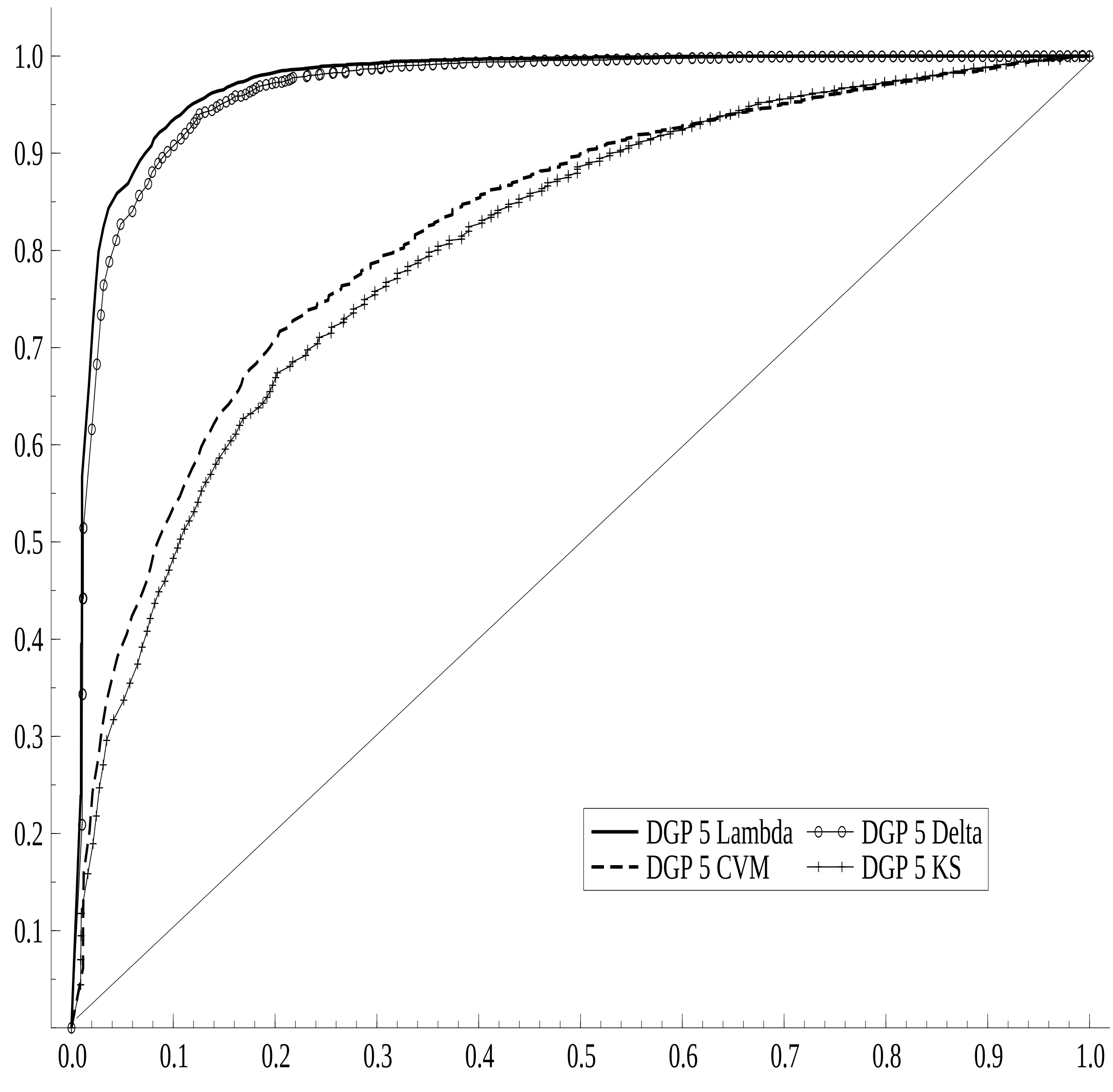

Figure 5: Size–Power Curves. DGP 5.300 Observations

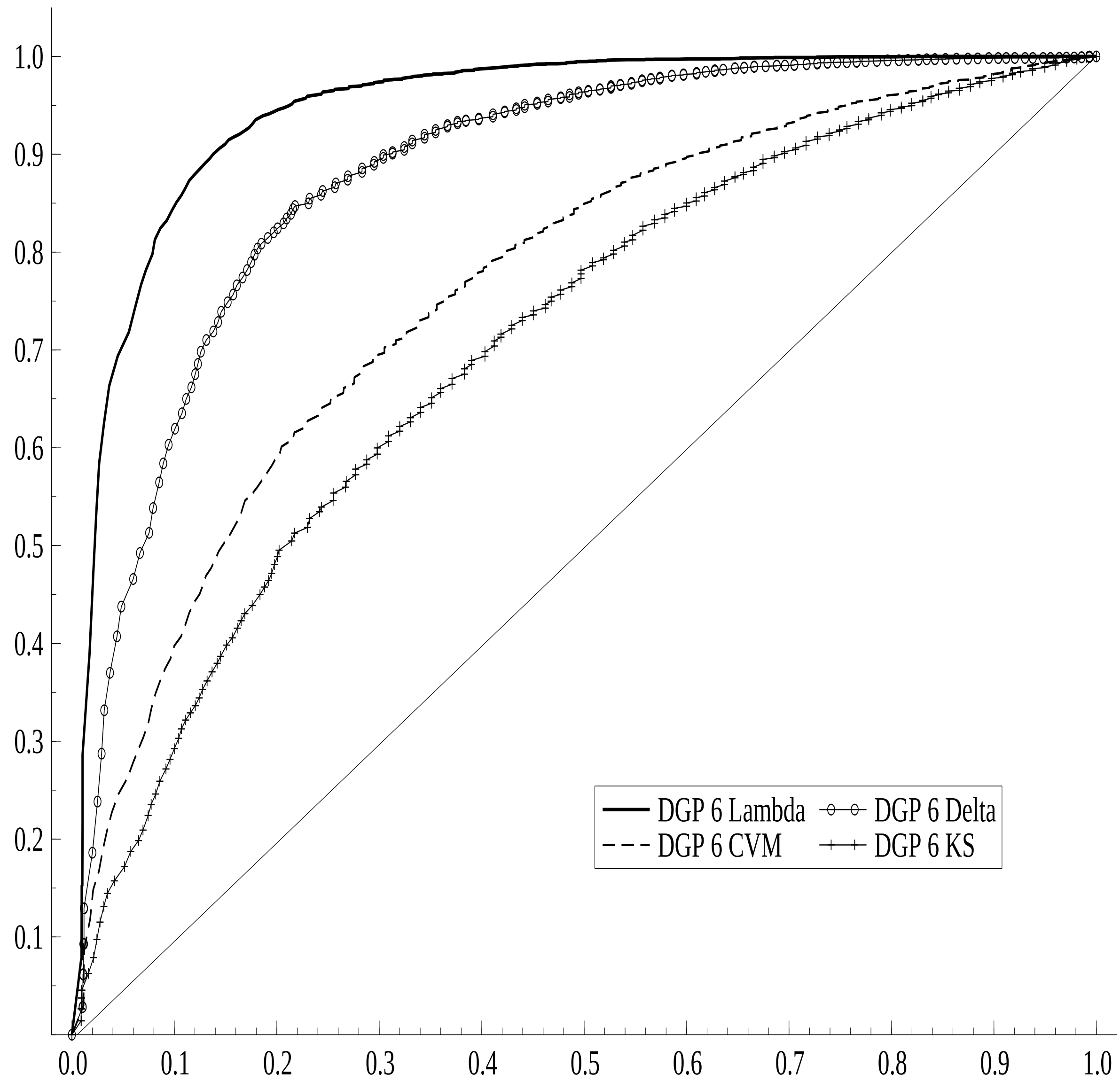

Figure 6: Size–Power Curves. DGP 6.300 Observations

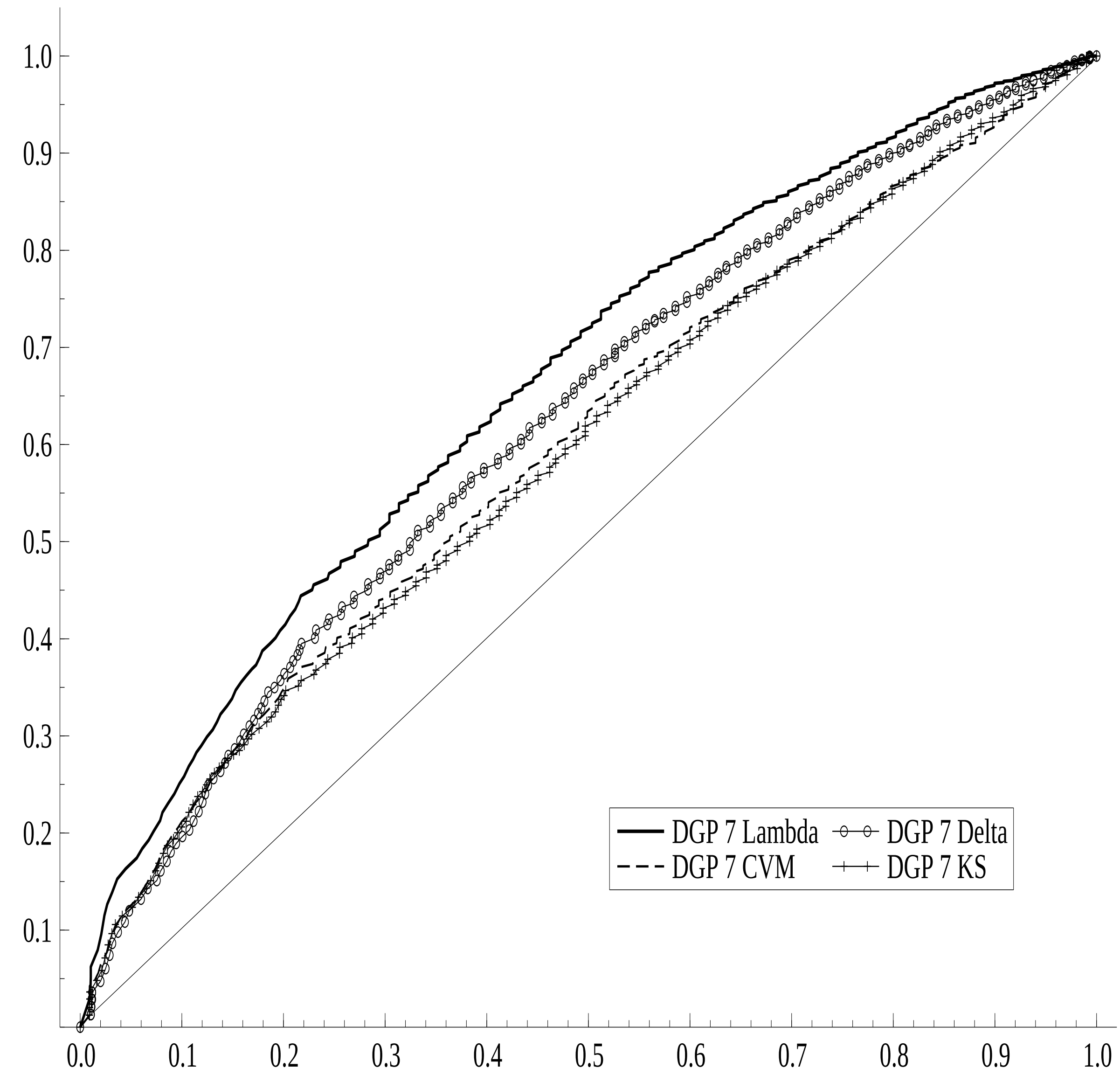

Figure 7: Size–Power Curves. DGP 7.300 Observations

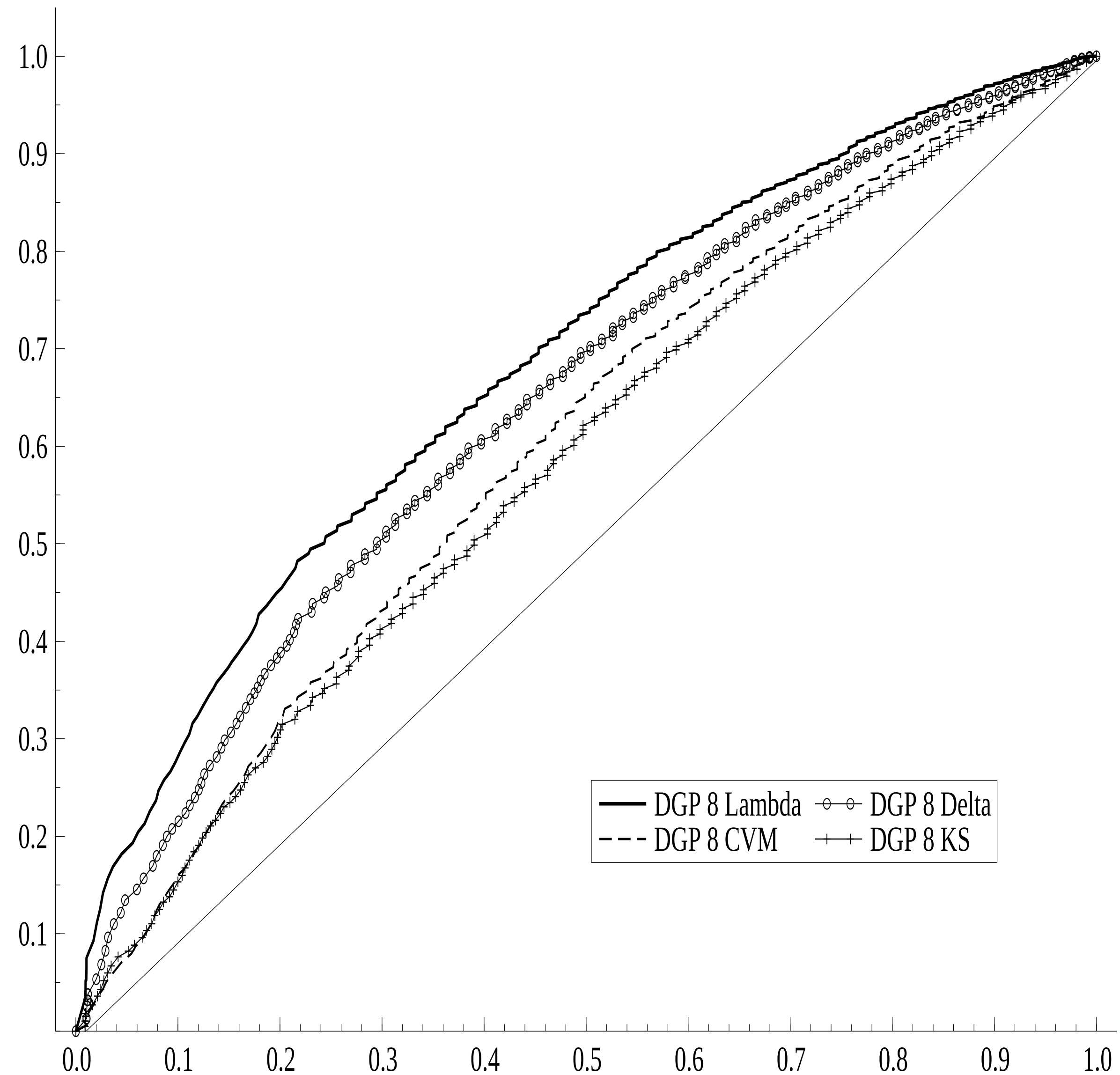

Figure 8: Size–Power Curves. DGP 8.300 Observations

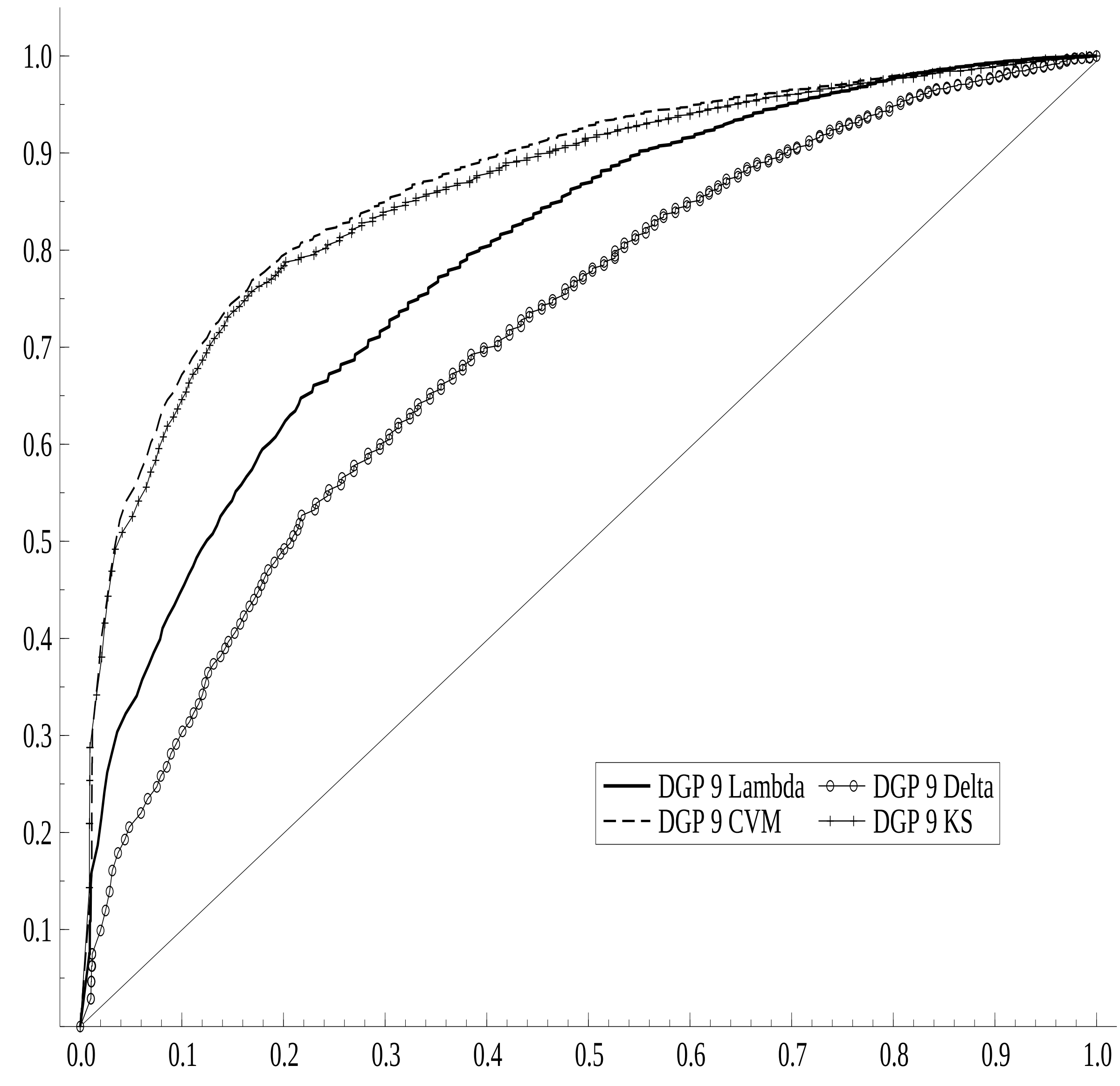

Figure 9: Size–Power Curves. DGP 9.300 Observations

Application to financial time series:

We consider financial time series: FX rates, indices, returns on equities in the technological and the banking sector:

Evidence for the presence of changes in the volatility processes can be deduced by a semiparametric analysis of the long-range dependence properties of the series of absolute returns $|R_t|$.

Comparison between the estimated scaling parameter obtained with

- the local Whittle estimator; see Robinson (1995),
- the estimator based on the wavelets; see Abry, Flandrin, Taqqu, Veitch (2002),

Both estimators are based on the assumption that the spectrum of a long–memory process has the form:

$$f(\lambda) \sim c_f \lambda^{-\vartheta}, \quad \lambda \to 0_+,$$

Unlike the local Whittle estimator, the wavelet estimator of the scaling parameter is unaffected by changes in the mean of the series, Veitch and Abry (1999): estimator of ϑ, which uses the independence properties of the wavelet coefficients $d_x(j,k)$ for fractional Gaussian noise and related LRD processes.

The wavelets coefficients are defined as

$$d_x(j,k) = \langle x, \psi_{j,k} \rangle,$$

where $\psi_{j,k}$ is a family of wavelet basis functions $\{\psi_{j,k} = 2^{-j/2}\psi_0(2^{-j}t - k\}$, $j = 1, \ldots, J$ are the octaves or scales, $k \in \mathcal{Z}$, ψ_0 is the mother wavelet, which has $\mathcal{N}$ zero moments, with $\mathcal{N} \geq 1$, i.e.,

$$\int t^k \psi_0(t) dt \equiv 0, \quad k = 0, \ldots, \mathcal{N} - 1.$$

By construction, the family of wavelet basis functions is scale invariant:

$$E d_x(j,\cdot)^2 = 2^{j\vartheta} c_f C, \quad \text{with} \quad C = \int |\lambda|^{-\vartheta} |\Psi_0(\lambda)|^2 d\lambda,$$

where $|\Psi_0(\lambda)|$ is the Fourier transform of the mother wavelet ψ_0.

The scaling parameter ϑ is estimated from the slope of the linear regression

$$\log_2 \left(E d_x(j,\cdot)^2 \right) = j\vartheta + \log_2(c_f C).$$

We use the Daubechies wavelets, with $\mathcal{N} = 4$.

The wavelet estimator has approximately the following asymptotic distribution

$$\sqrt{n}(\hat{\vartheta} - \vartheta) \sim N\left(0, \frac{1}{\ln^2(2)2^{1-j_1}}\right),$$

where j_1 is the lowest octave, the LRD behavior being captured by the octaves larger than j_1.

Local Whittle estimator: replace the expression of the spectrum by its approximation in the Whittle estimator:

$$\hat{\vartheta} = \arg\min_{\vartheta} \left\{ \ln\left(\frac{1}{m} \sum_{j=1}^{m} \frac{I(\lambda_j)}{\lambda_j^{-\vartheta}} \right) - \frac{\vartheta}{m} \sum_{j=1}^{m} \ln(\lambda_j) \right\},$$

$I(\lambda_j)$ is the periodogram evaluated on a set of m Fourier frequencies $\lambda_j = \pi j/T, j = 1, \ldots, m \ll [T/2]$,

Bandwidth parameter satisfies: $1/m + m/T \to 0$ as $T \to \infty$.

Under appropriate conditions, i.e;, differentiability of the spectrum near the zero frequency and existence of a moving average representation, the estimator has the following asymptotic distribution

$$\sqrt{m}(\hat{\vartheta} - \vartheta) \xrightarrow{d} N(0, 1),$$

Estimates of the scaling parameter ϑ. Standard errors in parentheses.

	Local Whittle	Wavelet
GBP–USD	0.5916 (0.1387)	0.1260 (0.0919)
S&P 500	0.6048 (0.0619)	0.1451 (0.0919)
FTSE 100	0.6378 (0.1222)	0.1036 (0.0919)
NASDAQ	0.5875 (0.0816)	0.0927 (0.0919)
IBM	0.3628 (0.1104)	0.1770 (0.0919)
Dell	0.7470 (0.1387)	0.0200 (0.0919)
Oracle	0.6012 (0.1139)	0.1021 (0.0919)
SAP	0.5017 (0.1231)	0.3747 (0.0919)
Microsoft	0.6407 (0.1291)	0.1528 (0.0919)
BNP	0.3585 (0.1260)	0.1941 (0.0919)
BoA	0.2230 (0.0814)	0.1862 (0.0919)
Citygroup	0.6224 (0.1240)	0.2047 (0.0919)
HSBC	0.3122 (0.0962)	0.1980 (0.0919)
Lloyds	0.7470 (0.1387)	0.1366 (0.0919)
JPM	0.5174 (0.1104)	0.1332 (0.0919)

- Estimated scaling parameter with the wavelet estimator is lower than the one obtained with the local Whittle estimator.
- This indicates the presence of change-points in volatility that "fool" the local Whittle estimator,
- We worked with the series of log returns $R_t = 100 \ln(P_t/P_{t-1})$, where P_t denotes the asset price at time t.
- We estimated the following GARCH(1,1) model on the series $\{R_t\}$:

$$
\begin{aligned}
R_t &= \mu + \psi\varepsilon_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_t^2), \\
\sigma_t^2 &= \omega + \beta\sigma_{t-1}^2 + \alpha\varepsilon_{t-1}^2.
\end{aligned}
$$

- Estimation results confirm that the tests are able to detect the presence of change-points in the volatility processes