Testing for Bubbles and Change–Points

Alan Kirman and Gilles Teyssière[1]

Journal of Economic Dynamics and Control
(2005), vol 29, 765-799. DOI:10.1016/j.jedc.2004.01.005

Abstract

A model for a financial asset is constructed with two types of agents, who differ in terms of their beliefs. The proportion of the two types changes over time according to stochastic processes which model the interaction between the agents. Agents do not persist in holding "wrong" beliefs and bubble–like phenomena in the asset price occur. We consider tests for detecting bubbles in the conditional mean and multiple changes in the conditional variance of the process. A wavelet analysis of the series generated by our models shows that the strong persistence in the volatility is likely to be the outcome of a mix of changes in regimes and a moderate level of long–range dependence. These results are consistent with what has been observed by Kokoszka and Teyssière (2002) and Teyssière (2003).
Keywords: Market interaction, bubbles, long–memory heteroskedasticity, pseudo long–memory, change–point, wavelets.
JEL Classification: C52, C22, D40, G12.

[1]Corresponding author. Laboratoire de Statistique Appliquée et MOdélisation Stochastique (SAMOS), Université Paris 1 – Panthéon–Sorbonne. E–mail: stats@gillesteyssiere.net http://www.gillesteyssiere.net
This is the outcome of a long lasting project which started with Kirman (1991), "Testing for Bubbles", and then supersedes this former paper. This version of the paper has been largely rewritten in Spring 2002 during the visit of Teyssière to the Department of Mathematics and Statistics of Utah State University, that he wishes to thank as well as Piotr and Gudi Kokoszka for their hospitality.

We are greatly indebted to Lajos Horváth and Piotr Kokoszka, with whom joint works on the issue of change–point tests for ARCH/GARCH processes strongly influenced the final version of this paper, to Clive Granger for his probing questions on the statistical properties of our processes, to Murad Taqqu and Patrice Abry who suggested the use of the wavelet analysis, to Remis Leipus and Alfredas Račkauskas for long discussions on the issue of bubbles testing, and Hans Föllmer with whom one of the stochastic models was developed. We would like to thank Raj Bhansali, George Evans, Liudas Giraitis, Cars Hommes, Davar Khoshnevisan, Danny Quah, Tom Sargent, Leigh Tesfatsion, Michael Woodford, two careful and constructive referees, and participants in the International Conference on Long-Range Dependent Stochastic Processes and their Applications, Bangalore, January 2002, in the colloquium of the Department of Mathematics and Statistics of Utah State University, and in the Stochastics seminar at the Department of Mathematics of the University of Utah, May 2002, for comments and suggestions.

I can calculate the motions of heavenly bodies but not the madness of people.
Isaac Newton

1 Introduction

Although the presence of "bubbles" and "herding" behaviour in financial markets is widely accepted there are still few theoretical models which generate such phenomena. Yet they are widely held to be responsible for the instability of foreign exchange markets for example. Indeed, "price bubbles" in markets although particularly associated with the markets for financial assets, have been documented for a wide variety of markets over a considerable period of time. One of the earliest bubbles was that in the price of red mullet in the first century A.D. The red mullet fever is documented by Cicero, Horace, Juvenal and Martial. A survey of other historical bubbles, such as the Tulip, South Sea and Mississippi bubbles, may be found in Garber (2000). More recently there has been a substantial literature on the theoretical basic for and testing of bubbles, see for example Blanchard and Watson (1982), Flood and Garber (1980), Meese (1986), Tirole (1985), West (1988), Woo (1987), Stiglitz (1990), Flood and Hodrick (1990), Donaldson and Kamstra (1996), Avery and Zemsky (1998), Shiller (2000) and Wu and Xiao (2002). One argument that has been advanced is that self reinforcing swings of opinion cause departures from fundamentals, Shiller (1981) observes that attention seems to switch from one share to another in financial markets without any particular change in the fundamentals associated with the share in question. Yet as the title of Shiller's (2000) book, "Irrational Exuberance" suggests, the implication is that such behaviour is irrational.

The purpose of this paper is to suggest that this is not the case. We will analyse a situation in which the participants in the market can choose between several forecasting rules. The nature of these speculative and self reinforcing rules will determine the demands of the various agents and determine the evolution of the equilibrium prices. We give a simple example in which people have a prospect of investing at home or abroad and they are influenced in their choices of rules and hence in their decisions by the yields obtained by their past choices and by, of course, the movements of the exchange rate. In this model self reinforcing changes in the exchange rate can occur, since as the number of individuals following a rule increases the success of that rule increases and more people tend to follow it. If the rule is an extrapolatory one then the exchange rate will leave its "fundamental" value and a "bubble" will occur. This sort of bubble is caused by switches in and transmission of expectation formation. For the reason mentioned this is self reinforcing and causes people to herd on one particular alternative type of forecast and eventually to switch back to another rule. In this case what is important is that there will be an important demand for the asset in question even if the underlying fundamentals do not seem to justify this.

In switching in this way, market participants are not being irrational. They will have good reason to focus on one opinion, one share or one currency for a period and then shift to another and a model of a stochastic process which results from such behaviour is proposed. Thus, it is the shifting composition of expectations that drives asset movements, in our case the exchange rate, and this is of course, at variance with the standard model in which expectations are homogeneous and rational and, of course, where no trade takes place. This is at variance with simple empirical observation of the spot market for foreign exchange where approximately $1.2 trillion per day was traded in 2001, for example.

As Bacchetta and van Wincoop (2003) point out, the obvious explanation lies in the heterogeneity of the agents on the market and, in particular, in the heterogeneity of their expectations. In the standard "representative agent" model there is no place for such heterogeneity and many authors have suggested that this is the reason for the poor predictive power of such models; evidence for the latter is given by Meese and Rogoff (1983), Frankel and Rose (1995) and Cheung *et al.* (2002). Furthermore, empirical observations suggest that expectations of actors on financial markets are indeed heterogeneous; see Chionis and MacDonald (2002).

A number of authors have introduced heterogeneous expectations into markets in different ways. One idea is simply to introduce agents who systematically have "wrong" expectations but who may survive nevertheless, such models were pioneered by De Long *et al.* (1989) who introduced "noise traders". Such a solution to the problem is not very appealing and if one takes account of the idea that agents may learn it is difficult to accept that certain actors will persist in their error. Another alternative is to introduce dispersed information into the model and one approach suggested by Townsend (1983) is to have symmetrically dispersed information and to analyse the consequences of "higher order expectations", i.e., expectations about others expectations. The idea here is that a small amount of non–fundamental trade may generate considerable volatility since traders perceive movements in asset prices as conveying information about future values of fundamentals; see Allen *et al.* (2003). Again, despite the more sophisticated reasoning attributed to agents, a certain degree of irrational behaviour is needed to generate the results.

Up to this point we have mentioned bubbles without being very precise about what we mean. In fact, there are two basic problems involved in the discussion of bubbles, on the one hand their definition, and on the other their detection and identification. In this paper we will present several variants on an economic model in which expectations can be heterogeneous, where agents learn from their experience of using different rules, how to form their expectations, where agents are not systematically wrong, and where departures from fundamentals can occur but where there is always a return to fundamentals. Thus we construct a model with reasonable properties which generates "bubbles" and then examine how various of the tests proposed for detecting bubbles perform on the data generated by the models.

These processes produce the bubble–like phenomena resulting from agents changing their forecasts. In introducing bubbles we follow Evans (1991), with two differences. Firstly, instead of simply testing data from a stochastic process with bubble-like characteristics, we use data from a model of economic behaviour with interacting agents. Secondly, this model is characterized by switches from one type of stochastic process to another, and is a particular instance of a random change–point process.

This lead us to consider here the statistical methods for the detection of changes in regimes. Recent research work by Kokoszka and Leipus (1999, 2000), Horváth, Kokoszka and Teyssière (2001, 2004), Mikosch and Stărică (1999), Berkes, Gombay, Horváth and Kokoszka (2004), Kokoszka and Teyssière (2002) and Berkes, Horváth and Kokoszka (2003) on the issue of change–point testing for the class of GARCH processes motivated preliminary studies by Teyssière (2003) and Kirman and Teyssière (2002b) which show that these tests detect switches in the volatility process for this class of microeconomic models, and further can be used for the detection of bubbles. The relevance of the use of this class of change–points tests is reinforced by Mikosch's works (1999), which revealed that changes in the volatility of asset prices match macroeconomic switches from expansion to recession and vice versa. It then appears that the phenomena of bubbles in asset prices, persistence and changes in their volatility might be closely related.

This is of practical interest since the phenomena of persistence and changes in volatility can be formally defined and then detected by statistical methods, while there is no formal definition for bubbles which makes their detection more problematic.

We will require two features of bubbles which go beyond the simple departure from fundamentals. Firstly, they should "burst" at some time and not be perpetually explosive, see Diba and Grossman (1988), and secondly that they should be endogenous, i.e., not directly produced by exogenous shocks.

The earlier part of the literature on bubbles, such as the contributions of Le Roy and Porter (1981), Shiller (1981) and Blanchard and Watson (1982), all came to the view that asset prices were too volatile to be explained by fundamentals alone. Thus it was argued that there was "excess volatility". Meese and Rogoff (1983) came to the same conclusion for exchange rates.

The debate has, however, swung somewhat in the opposite direction. In part it has been suggested that the econometric analysis in the papers mentioned was faulty and in part that the process governing the fundamentals had been misspecified, a good idea of the main issues in this discussion can be obtained from Campbell and Shiller (1987), Mankiw, Romer and Shapiro (1985), Marsh and Merten (1986), West (1987, 1988), Flood and Hodrick (1990) and Donaldson and Kamstra (1996), or even that some unobserved fundamentals might have been omitted, see Hamilton and Whiteman (1985). Alternatively as suggested by Miller (1991) even small changes may bring about large shifts in prices if horizons are sufficiently long. Two other facets of the debate have also developed: that based on the psychology of investors' behaviour, see Wärneryd (2001), and that which shows that bubbles will occur, even for assets with perfectly defined fundamental values in experimental markets, see Caginalp *et al.* (1998).

Diba and Grossman (1988) claimed that the data for stock prices does not have the explosive characteristics one would expect if bubbles existed. Perron (1989) however, suggested that the unit root tests commonly used may fail to reject the presence of unit roots, when in fact the underlying process is one with a "broken trend" or a shift in regime. Indeed Evans (1991) has found, by testing data from simulating a stochastic process known to contain bubbles, that in general the unit roots hypothesis was not rejected. Recently, Wu and Xiao (2002) have shown that these procedures based on unit root tests are also unable to detect collapsible bubbles.

The real question seems, however, not to be quite as simple as that discussed in the general debate on the issue. If asset prices do in general follow fundamentals, but randomly depart from them, then the situation is rather complicated. If fundamentals follow a random walk as theory might suggest in the case of stock prices for example, then for some, maybe substantial, part of the time this is the process that will be realised. The process that is followed at other periods has, of course, to be specified.

In particular, it should be clear that the basic point at issue here is not does or does not the asset price process have unit roots but how far and for how long does it deviate from that process and how can one separate out these deviations? In the long run, as indeed the word suggests, bubbles do not matter, but their impact in the short run may be very significant.

In the foreign exchange market which we take as an example, roughly two thirds of all turnover consists of spot transactions. Since dealers have very short horizons, many have to have a closed position at the end of the day, and their customers are sensitive to price changes, it is clear that episodes in which extrapolatory behaviour can take the market away from fundamentals can be very important. Yet most dealers argue that, "in the long run fundamentals matter", see Barrow (1994). The way in which fundamentals eventually pull prices back is through underlying order flows, see Kouri (1983) and Lyons (2001), but since these in turn are affected by the evolution of current prices the magnitude and duration of deviations are difficult to calculate.

We consider here a class of models which has the basic characteristics just outlined. This class of microeconomic models generating non–homogeneous processes is of interest in financial econometrics as it constitutes a microeconomic–based framework for the analysis of long–range dependence in asset price volatility. These empirical volatility processes are characterized by a slow hyperbolic decay of the autocorrelation function and a singularity of their spectrum near the zero frequency, properties which are typical of long–range dependent, henceforth LRD, processes, also called strongly dependent or long–memory processes. However, two features lead us to conclude that volatility processes are more complex than standard LRD processes.

Volatility processes are not trended and then differ from the most common LRD process, the fractionally integrated $I(d)$ process, see Granger (2002) for further details. Thus, volatility processes might be the realisation of either a more sophisticated LRD process, see e.g., Teyssière (1996), or a process which spuriously exhibits the properties of LRD processes mentioned above. In Kirman and Teyssière (2002b) we called these latter processes “pseudo long–memory processes”. A process of interest is the non–homogeneous GARCH volatility process, i.e., a GARCH process with coefficients constant only on a time interval of finite length. Mikosch and Stărică (1999) have demonstrated that the power transformation of a non–homogeneous GARCH(1,1) process, with coefficients changing so that the unconditional variance of the process is not constant, displays strong dependence, the intensity of which is proportional to the magnitude of the change in the unconditional variance. Evidence reported in Teyssière (2003) show that R/S–type tests for stationarity against LRD alternatives wrongly detect LRD–type behaviour on power transformations of non–homogeneous GARCH processes, while the power of these tests tend to their size when the unconditional variance of the process is constant. Thus, the empirical properties of asset price volatility might be a statistical artefact, and all usual conclusions in the financial econometric literature on the presence of strong dependence in asset price volatility might be simply the consequence of the inadequacy of the standard statistical procedures for adjudicating between genuine long range dependence and switches in regimes.

A multivariate analysis of the empirical LRD properties pioneered by Teyssière (1997, 1998) show that the volatility and co–volatility of some financial time series share a common intensity of strong dependence. Since it is likely that financial time series are affected by common breaks, common breaks in the volatility processes might be the cause of the empirically observed commonality of the strong dependence in the volatility process; see Teyssière (2003) and the last section of Granger and Hyung (1999). Following a personal communication from Murad Taqqu and Patrice Abry, Teyssière (2003) and Kokoszka and Teyssière (2002) conducted a wavelet based analysis of the LRD properties of asset prices empirical volatilities and co–volatilities, since the multi-resolution principle of wavelets analysis makes this method robust to un–periodic changes in mean of the process. Simulated and empirical results reported in Teyssière (2003) and Kokoszka and Teyssière (2002) suggest that the volatility and co–volatility processes of asset prices returns mix change–point and LRD processes with a moderate level of persistence.

This paper is organized as follows. The class of economic models generating these bubbles are introduced in section **2**. The opinion diffusion processes are detailed in section **3**. In section **4**, we discuss the LRD and change–point processes alternatives and introduce the wavelet analysis for adjudicating between them. The testing procedure for the detection of collapsible bubbles and change–points, with the simulation results are presented in section **5**. Section **6** concludes.

2 Economic model with heterogeneous agents

This economic models, presented here are variants on that developed in its operational form by Kirman and Teyssière (2002a, 2002b) and Teyssière (2003). It has the essential feature that the price process departs from fundamentals and then returns to them. In addition, bubbles can be both negative and positive. Models which are close in spirit to this sort of model are those of Brock and Hommes (1999) and LeBaron (2001).

We now give a brief description of the model. The different variants of which are then simulated to generate the data to be used for evaluating the recent testing procedures for bubbles in levels and change–points in volatility or their absence.

Agents are faced with a price process P_t for a financial asset and form expectations about tomorrow's prices. There are two different rules for forming expectations and each agent uses one of them. However, the expectations of the individual agents will change over time as they decide which forecasting rule to adopt. In the first case they are influenced by random meetings with other agents. In the other case they will base their judgment on the basis of the success of the rule. Lastly they may have a combination of the two. Call the two methods of forming expectations the two "opinions" in the model. These could be founded on the forecasts given by "gurus" who offer their advice or could be considered as being reasonable rules from which the agents choose. There is nothing in our model which would exclude having any finite number of such rules. In the case that we consider, that with two possible rules, then if there are N agents, we say that

Definition 1 *The state of the system at time t is defined by the number k_t of agents holding opinion one, i.e., $k_t \in \{0, 1, \ldots, N\}$.*

The processes governing the evolution of the opinion diffusion process $\{k_t\}$ will be introduced in section 3.

Consider two types of individuals who forecast the value of an asset or, as in the model developed by Frankel and Froot (1986), the value of the exchange rate.

"Fundamentalists" believe that the exchange rate P_t at time t is related to some underlying fundamental $\bar{P}_t$ which might be a constant $\bar{P}$, some long run equilibrium, or might be governed by some dynamic deterministic or stochastic process. We consider here a standard random walk process for $\bar{P}_t$

$$\bar{P}_t = \bar{P}_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2). \tag{1}$$

Fundamentalists' forecast for the value at the next period, conditional on the information set I_t available at time t, is given by

$$E^f\left(P_{t+1}|I_t\right) = \bar{P}_t + \sum_{j=1}^{M^f} \nu_j (P_{t-j+1} - \bar{P}_{t-j}), \tag{2}$$

where ν_j, $j = 1, \ldots, M^f$ are positive constants, M^f is the memory of the fundamentalists. In most of the early models the memory of the fundamentalist and of the chartist was limited to one period.

Chartists, on the other hand, forecast by simple extrapolation of the past history of prices and hence predict that the next period exchange rate will be given by

$$E^c\left(P_{t+1}|I_t\right) = \sum_{j=0}^{M^c} h_j P_{t-j}, \tag{3}$$

where h_j, $j = 0, \ldots, M^c$ are constants, M^c is the memory of the chartists. It would, of course, be interesting to try other more sophisticated forms of extrapolation. These would change the forecasts of the chartists but would not change the basic stochastic alternation between regimes for the case of epidemiologic opinion diffusion.

The market view or forecast is given by a weighted average of the two forecasts, i.e.,

$$E^m(P_{t+1}|I_t) = w_t E^f(P_{t+1}|I_t) + (1 - w_t)E^c(P_{t+1}|I_t), \tag{4}$$

where w_t is the proportion of fundamentalists. The weights are determined in Frankel and Froot's model by portfolio managers who effectively choose the weights in such a way as to make the actual outcome constant with the market forecast.

In the model considered here the weights w_t are determined endogenously, and they reflect the number of agents who act according to each view. To see this consider the process as taking place in two steps. Firstly, an individual meets another and forms an opinion after this meeting about how prices will change. "Meeting" of course does not mean meeting in the literal physical sense, a better word might be "contact". Dealers on markets receive and transmit information in several ways and each of these contacts could be a meeting in our sense.

A market based purely on private information would not be totally realistic and it is often the case that dealers take account of "market sentiment". This is in line with Keynes' observation that it is better to be wrong with the crowd than wrong on your own. Thus we introduce a noisy market signal as an indicator of how many people hold each opinion.

The process can be described as follows:

1. Agents meet each other at random and are converted to each others' opinions as defined in the process described above. Consider k_t as the number of individuals at time t who are "fundamentalists" and the remaining $N - k_t$ as "chartists". Allow some fixed number M of meetings to take place at each time.

2. Defining $q_t = k_t/N$ each agent now makes an observation of q_t i.e., tries to assess which opinion is in the majority. She observes q_t with some noise. Thus, the signal she receives is
$$q_{i,t} = q_t + \varepsilon_{i,t}, \quad \varepsilon_{i,t} \sim N(0, \sigma_q^2), \quad q_{i,t} \in [0, 1]. \tag{5}$$
If now agent i receives a signal $q_{i,t} \geq 1/2$, then she will make a fundamentalist forecast since the majority is doing so. Conversely, if $q_{i,t} < 1/2$ she will forecast as a chartist and act accordingly. The number and proportion of agents who base their demand on fundamental forecasts are therefore given by
$$w_t = N^{-1} \sum_{i=1}^{N} \# \left\{ i : q_{i,t} \geq \frac{1}{2} \right\}. \tag{6}$$

As one of the requirements at the outset we wish agents to take account of both local and global information. In fact the outcomes in our model will depend on the relation between the two. For a discussion of similar problems see Ellison and Fudenberg (1993). The relationship between the processes $\{w_t\}$ and $\{k_t\}$ depends on the variance of the noise in the signal about the majority. If this variance is large, the signal is of little value and people will choose more or less at random. In this case, the process $\{w_t\}$ will spend more time close to the equal proportion situation than does the $\{k_t\}$ process. It seems therefore, reasonable to consider the variance as small since otherwise agents should give more weight to their private information. Having now determined the proportion of agents who forecast as fundamentalists the market forecast is given by equation (4) and the market forecast of its change by

$$\Delta^m P_{t+1}|I_t = E^m(P_{t+1}|I_t) - P_t. \tag{7}$$

The price on the market, i.e., the market exchange rate if one is considering the market for foreign exchange, is given by

$$P_t = cE^m(P_{t+1}|I_t) + Z_t, \tag{8}$$

where c is a constant and Z_t is an index of a vector of fundamental variables according to Frankel and Froot (1986). Can this model be derived from individual behaviour? To see that the answer is positive consider the following simple model, suggested by Michael Woodford: agent i has a utility function given by

$$U^i(W^i_{t+1}) = E(W^i_{t+1}) - \lambda \mathrm{Var}(W^i_{t+1}), \tag{9}$$

where λ denotes the risk aversion coefficient, $E(.)$ denotes the expectation operator and W^i_{t+1}, her wealth at time $t+1$, is given by

$$W^i_{t+1} = (1+\rho_{t+1})P_{t+1}d^i_t + (W^i_t - P_t d^i_t)(1+r), \tag{10}$$

where the variables are defined as follows

- ρ_{t+1} is the dividend in foreign currency paid on one unit of foreign currency,
- P_{t+1} is the exchange rate at $t+1$,
- d^i_t is the demand by the i^{th} individual for foreign currency,
- r is the interest rate on holdings of domestic currency.

The variables P_{t+1} and ρ_{t+1} are both considered by agents to be random variables. The first two moments of the distribution of P_{t+1}, from the point of view of individual i, are given by

$$E(P_{t+1}) = \Delta P^i_{t+1} + P_t, \quad \mathrm{Var}(P_{t+1}) = \sigma^2_P, \tag{11}$$

and for ρ_{t+1} by

$$E(\rho_{t+1}) = \rho, \quad \mathrm{Var}(\rho_{t+1}) = \sigma^2_\rho. \tag{12}$$

The variance of P_{t+1} is, in reality, time dependent, but we are following in the tradition now established in the literature of allowing some error of perception by our agents who consider that the conditional variance is constant. Furthermore, assume ρ_{t+1} and P_{t+1} to be independent. From these assumptions

$$E(W^i_{t+1}|I_t) = (1+\rho)E^i(P_{t+1}|I_t)d^i_t + (W^i_t - P_t d^i_t)(1+r), \tag{13}$$

and

$$\mathrm{Var}(W^i_{t+1}|I_t) = (d^i_t)^2\zeta_t \quad \text{with} \quad \zeta_t = \mathrm{Var}\left(P_{t+1}(1+\rho_{t+1})\right). \tag{14}$$

Demand d^i_t is found by maximising utility and writing the first order condition

$$(1+\rho)E^i(P_{t+1}|I_t) - (1+r)P_t - 2\zeta_t\lambda d^i_t = 0, \tag{15}$$

where $E^i(.|I_t)$ denotes the expectation of an agent of type i. Since w_t is the proportion of fundamentalists at time t, the market demand is then given by:

$$d_t = \frac{(1+\rho)\left(w_t E^f(P_{t+1}|I_t) + (1-w_t)E^c(P_{t+1}|I_t)\right) - (1+r)P_t}{2\zeta_t\lambda}. \tag{16}$$

The supply of foreign exchange is given by X_t and since agents only differ in their forecasts as to the value of the future exchange rate, then the market is in equilibrium if $X_t = d_t$, which gives

$$(1+r)P_t = (1+\rho)\Big(w_t E^f(P_{t+1}|I_t) + (1-w_t)E^c(P_{t+1}|I_t)\Big) - 2\zeta_t\lambda X_t. \tag{17}$$

which is of the form of equation (8) with c and Z_t defined appropriately.

A standard condition for a solution to equation (17) is

$$\frac{(1+r)}{(1+\rho)} \neq \Big(w_t \sum \text{coefficients of } P_t \text{ in } E^f(P_{t+1}|I_t) + (1-w_t)\sum \text{coefficients of } P_t \text{ in } E^c(P_{t+1}|I_t)\Big), \tag{18}$$

for all $w_t \in [0,1]$.

Clearly the “fundamental” value $\bar{P}_t$ should be linked to X_t and we suppose here that $2\zeta_t\lambda X_t/(1+\rho) = \varpi\bar{P}_t$. We are in effect considering here that X_t is a “liquidity demand”, i.e., not one generated by speculative motives and therefore exogenous to the model. If we assume that $M^f = M^c = 1$, then the equilibrium price is given by

$$P_t = \frac{w_t - \varpi}{A}\bar{P}_t - \frac{w_t\nu_1}{A}\bar{P}_{t-1} + \frac{(1-w_t)h_1}{A}P_{t-1}, \quad A = \frac{1+r}{1+\rho} - (1-w_t)h_0 - w_t\nu_1. \tag{19}$$

We set $\varpi = 1 - (1+r)/(1+\rho)$, so that if $\bar{P}_t = \bar{P}$, $\forall t$, then when $w_t = 1$, i.e., all agents are fundamentalists, $P_t = \bar{P}$.[1]

To come back to our model, when $\{w_t\}$ switches from 0 to 1 and vice versa, equation (19) defines a change–point process in the conditional mean. However, Figures 1 and 2 in the appendix, which refer to different processes $\{w_t\}$, show that this process never herds on the extremes 0 and 1, and sometimes moves gradually. Thus, equation (19) defines a process more complex than a pure change-point process in the conditional mean. We wish to test whether this varying regime process generates the empirical properties of volatility processes. As shown by equations (5) and (6), the process $\{w_t\}$ is the composition of the process $\{k_t\}$ and a majority process, i.e., the “beauty queen” idea by Keynes. The herding behaviour of the process $\{w_t\}$ is controlled by the parameter σ_q in equation (5) and the parameters governing the herding behaviour of the opinion diffusion process $\{k_t\}$.

3 Opinions diffusion processes

3.1 Epidemiologic diffusion

In a first approach, we consider that the process $\{k_t\}$ follows an epidemiologic diffusion process developed by Hans Föllmer and originally based on experimental evidence on the behaviour of

[1] It is of interest to link our model to that of the standard Rational Expectation model; see McCallum (1983). It should be noted that our model differs from the standard case examined by George Evans (1986) in his pioneering article on rational bubbles. We thank an anonymous referee for these observations.

ants and considered by Kirman (1991, 1993). The stochastic process governing the state process $\{k_t\}$ evolves as follows. Two agents meet at random and the first is converted to the second's view with probability $(1-\delta)$. If the meeting is considered as a drawing from an urn with balls of two different colours, it is obvious that which agents is the "first" and which is the "second" is of no importance since the symmetric event occurs with the same probability. There is also a small probability ε that the first agent will change her opinion independently of whom she meets. This is a technical necessity to prevent the process from being "absorbed" into one of the two states 0 or N, but can be allowed to go to zero as N becomes large. This ε can be thought of as the replacement of some old agents in each new period by agents who may hold either opinion, or by some external shock which influences some people's expectations. Indeed in what follows we shall require for the basic results that ε be small.

The process $\{k_t\}$ then evolves as follows

$$k \longrightarrow \begin{cases} k+1 \text{ with probability } \; p(k,k+1) = \left(1-\frac{k}{N}\right)\left(\varepsilon + (1-\delta)\frac{k}{N-1}\right), \\ k \text{ with probability } \; p(k,k) = 1 - p(k,k+1) - p(k,k-1), \\ k-1 \text{ with probability } \; p(k,k-1) = \frac{k}{N}\left(\varepsilon + (1-\delta)\frac{N-k}{N-1}\right). \end{cases} \tag{20}$$

The first problem is to look at the equilibrium distribution $\mu(k)$, $k = 0, \ldots, N$ of the Markov Chain defined by (20). This is important in the economic model since it describes the proportion of time that the system will spend in each state. The equilibrium distribution is given by

$$\mu(k) = \sum_{l=0}^{N} \mu(l) p(l,k), \tag{21}$$

but given that the process is symmetric and reversible then it follows that

$$\mu(l) p(l,k) = \mu(k) p(k,l). \tag{22}$$

From this expression one obtains

$$\frac{\mu(k+1)}{\mu(k)} = \frac{p(k,k+1)}{p(k+1,k)} = \frac{\left(1-\frac{k}{N}\right)\left(\varepsilon + (1-\delta)\frac{k}{N-1}\right)}{\frac{k+1}{N}\left(\varepsilon + (1-\delta)\left(1-\frac{k}{N-1}\right)\right)}, \tag{23}$$

since it is clear from (22) that

$$\mu(k) = \frac{\frac{\mu(1)}{\mu(0)} \cdots \frac{\mu(k)}{\mu(k-1)}}{1 + \sum_{l=1}^{N} \frac{\mu(1)}{\mu(0)} \cdots \frac{\mu(l)}{\mu(l-1)}}. \tag{24}$$

Now the form of $\mu(k)$ will depend, naturally, on the values of ε and δ. The case of particular interest here is that in which $\mu(k)$ has the form indicated in Figure 3 in the appendix. It is easy to see that if $\varepsilon < (1-\delta)/(N-1)$ then $\mu(k)$ will indeed be convex. Thus this case, in which the process spends most of its time in the extremes, corresponds to the case in which the probability of "self conversion" is small relative to the probability of being converted by the person one meets. Although this probability of conversion is independent of the numbers in each group, which type will actually meet which type depends on the relative numbers in each type at any moment, i.e. on the state of the system. Thus, when one type is in the minority conversion of any individual is much less likely than when the numbers of the two types are fairly equal.

The ε might be considered as being simply a technical artefact, therefore it is worth looking at what happens to the process when N becomes large and ε goes to zero. Consider the asymptotic form of $\mu(k)$ when we choose ε for each N so that $\varepsilon < (1-\delta)/N$. When N becomes large redefine μ as $\mu(k/N)$ and consider the limit distribution as $N \to \infty$. Call this limit distribution, which will be continuous, f. Then one can prove the following:

PROPOSITION 1 *f is the density of a symmetric Beta distribution, i.e., $f(x) = Cx^{\varsigma-1}(1-x)^{\varsigma-1}$, where C is a constant.*

This proposition was proved by Föllmer and the proof is given in Kirman (1993). For any given value of k, $\mu(k/N)$ increases proportionately with N. Thus, for $\varsigma < 1$ the distribution has the form illustrated in Figure 3.

This stochastic model of shifts of opinion given here is related to the urn models of Arthur *et al.* (1986) and also to models which have been developed for shifts in voter opinion; see the examples given by Weidlich, cited in Haken (1977), where a similar bimodal distribution is derived. The latter model could also have been taken as the basis for the conversion from one opinion to another here.

3.2 Learning processes

For the epidemiologic process developed in the previous section, the proportion of fundamentalists and the forecasts of agents do not depend on the past performance of forecasts functions. Teyssière (2003) introduced a diffusion process for k_t which depends on the accuracy of the forecast functions in the recent periods: the probability of choosing a particular forecast function depends on its comparative performance over the competing forecast function. We use Theil's (1961) U–statistic as measure of forecast accuracy over the last M periods

$$U_M^j = \sqrt{\frac{M^{-1}\sum_{l=1}^{M} w_l \left(P_{t-l} - E^j(P_{t-l}|I_{t-1-l})\right)^2}{M^{-1}\sum_{l=1}^{M} w_l P_{t-l}^2}}, \quad j \in \{c, f\}, \quad \sum_l w_l = 1, \tag{25}$$

M being the learning memory of agents, the weights $w_l, l = 1, \ldots, M$ representing the relative importance of the forecast errors at time $t-l$. We choose here an exponential choice function $g^j(\cdot)$ for the forecast function $E^j(\cdot)$, i.e.,

$$g^j(t) = \exp(-\varrho U_M^j), \quad \varrho > 0, \quad j \in \{c, f\}, \tag{26}$$

the parameter ϱ being the intensity of choice. At time t, agents will chose with probability $\pi^f(t)$ the fundamentalist forecast function, where

$$\pi^f(t) = \frac{g^f(t)}{g^f(t) + g^c(t)}, \tag{27}$$

the probability of choosing the chartist forecast function is then $\pi^c(t) = 1 - \pi^f(t)$. This choice mechanism is standard in the economic literature; see Aoki (1996). Brock and Hommes (1997) considered a similar mechanism for choosing between chartist and fundamentalist view, the choice being based on the comparison between the realized profits without normalising to a common scale.

3.3 Epidemiologic processes with learning

We assume here that agents keep a record of the past meetings with other agents, in particular they keep in memory the relevance of the opinions of the agents they met. If an agent's opinion in the previous meeting appeared to be the "right" one, then the probability of conversion to that agent's opinion is equal to $(1 - \delta_l)$, while if this opinion turned out to be "wrong", this probability of conversion is then equal to $(1 - \delta_h)$, with $\delta_h > \delta_l$. The herding behaviour of the process $\{k_t\}$ is controlled by the two parameters δ_h and δ_l. By "right", we mean in the simplest model that the forecast was closer to the realised value than that of the other opinion.

We extend this learning mechanism by taking into account the M previous meetings with the agent, e.g., the probability of conversion is equal $(1 - \delta_h^{1/M})$ if agent's views were wrong in the last M meetings, and is equal to $(1 - \delta_l^M)$ if agent's views were right in the last M meetings. We can see that

$$(1 - \delta_l^M) \to 1, \quad (1 - \delta_h^{1/M}) \to 0, \quad M \to \infty. \tag{28}$$

Generally, if agent was right υ times in the last M meetings, the probability that he converts another agent is equal to

$$\begin{aligned} 1 - \delta^\star &= 1 - \delta_h^{\left(\frac{1}{M-\upsilon}\right)} \delta_l^\upsilon \quad \text{if} \quad \upsilon < M, \\ &= 1 - \delta_l^M \quad \text{if} \quad \upsilon = M. \end{aligned} \tag{29}$$

We generalize this approach by considering that δ takes a continuum of values in $[0, 1]$, and make δ dependent on the magnitude of forecast accuracy U^i of agent i, i.e.,

$$\delta = \varkappa + \frac{\exp(-\eta U^i)}{\exp(-\eta U^c) + \exp(-\eta U^f)}, \quad \varkappa \geq 0, \quad \eta > 0, \quad \delta \in [\varkappa, 1], \quad i = 1, \dots, N, \tag{30}$$

where $\varkappa$ is the minimum value for δ, $U^i = U_M^c$ if agent i is chartist and $U^i = U_M^f$ if agent i is fundamentalist, U_M^j, $j \in \{c, f\}$, being defined by equation (25). This approach, like that defined in (27), can be derived from an "exploration"/"exploitation" trade–off, where agents weight their probabilities by their previous gains from an action and the information they could accumulate by trying other alternatives.

4 Long–range dependent vs. change–point processes

According to previous studies by Kirman and Teyssière (2002a, 2002b) and Teyssière (2003), the series P_t generated by the underlying Markov process described above have the following features: unit roots, heteroskedastic errors and a particular type of nonlinearity such that $\text{Cov}(|R_t|^n, |R_{t+j}|^n)$ decays slowly as $j \to \infty$, where $R_t = \Delta \ln(P_t)$ and $n > 0$, this decay being the slowest for $n = 1$. This strong dependence is explained by at least two classes of conditional variance processes: the LRD–ARCH processes and some non–homogeneous GARCH processes, i.e., the GARCH processes with parameters changing so that the unconditional variance of the process is not constant. This non–stationarity of the process might explain the behaviour of the sample behaviour and extremes of the correlations structure of financial time series, that cannot be captured by stationary GARCH(1,1) processes; see Mikosch and Stărică (2004, 2000).

The class of LRD–ARCH processes, introduced by Robinson (1991), extends the class of ARCH/GARCH models of Engle (1982) to the more general case of ARCH(∞) models. This class of models has been further developed by Granger and Ding (1995), Ding and Granger (1996) and other authors. The general form of a LRD–ARCH process is

$$R_t = m_R + \sigma_t \varepsilon_t, \quad \varepsilon_t \sim D(0,1), \quad \sigma_t^n = \omega + \varphi(L) g(\varepsilon_t), \tag{31}$$

where m_R denotes the regression function, $\varphi(L) = \sum_{i=1}^{\infty} \varphi_i L^i$ is an infinite order lag polynomial the coefficients of which are positive and have asymptotically the following hyperbolic rate of decay $\varphi_j = O\left(j^{-(1+\vartheta/2)}\right)$, where $\vartheta \in (0,1)$ is the scaling parameter, $g(\varepsilon_t)$ is a function of the innovations ε_t including non–linear transformations, $D(0,1)$ is a distribution with mean equal to zero and variance equal to one. With these assumptions, the autocorrelation function (ACF) of the sequence of squared returns satisfies

$$\mathrm{Cov}(R_t^2, R_{t+j}^2) = O(j^{\vartheta-1}), \tag{32}$$

where the scaling parameter ϑ governs the slow rate of decay of the ACF and then parsimoniously summarizes the degree of long–range dependence of the series. Interested readers are referred to Beran (1994) and Robinson (1994) for a complete reference on long–memory processes, and to Giraitis, Kokoszka and Leipus (2000), Kazakevičius and Leipus (2002, 2003), and Giraitis, Leipus and Surgailis (2003) for the memory properties of the class of ARCH(∞) processes. The strong dependence in the conditional variance can be alternatively modeled by the long-memory stochastic volatility processes, the memory properties of which have been investigated by Robinson (2001).

Although this class of LRD–ARCH processes is appealing, the empirical properties of volatility series are more complex than the ones of the simple LRD fractionally integrated $I(d)$ process, where the fractional differencing parameter d is related to the scaling parameter ϑ by $\vartheta = 2d$. An $I(d)$ process exhibits local trends, the slope of which is increasing with the parameter d, while the empirical volatility series $|R_t|^n$ are not trended. These features, generously pointed out to us by Clive Granger in a personal communication in September 2000, are detailed in Granger (2002). In fact, as explained in Kirman and Teyssière (2002a, 2002b) and Teyssière (2003), the volatility series generated by our model do not display a trend as well and resemble the volatility series of asset prices returns.

It is well known that statistical tests wrongly detect long–range dependence when the true process is a change–point process. In particular, according to Mikosch and Stărică (1999, 2004) long-range dependence in the volatility process can be spurious and the consequence of the concatenation of short-range dependent GARCH(1,1) processes with changing coefficients. A subsequent work by Stărică and Granger (2001) has shown that a shift in variance model has a better forecast performance for the volatility than an $I(d)$ process. There is a substantial literature on change–point analysis, see e.g., Csörgő and Horváth (1997), Basseville and Nikiforov (1993) and Brodsky and Darkhovsky (1993) for recent references, which however focuses on conditional mean processes. An account on the recent developments in the change–point detection literature is given in Kokoszka and Leipus (2003), while Račkauskas and Suquet (2003) provide a survey on the Hölderian principle for testing for epidemic changes.

Granger's remark lead us to consider the occurrence of change–points in the conditional variance as a possible alternative explanation of the empirical LRD properties. Since in our model asset prices P_t are a varying combination of the previous prices P_{t-1}, and the fundamentals $\bar{P}_t$ and $\bar{P}_{t-1}$, we were first interested in checking whether we can detect the changes in the proportion of fundamentalists w_t at time t, i.e., the swings in opinion, which generate the long-memory in the volatility process, as the degree of long–range dependence in the volatility of the models was linked to the parameters controlling the swings in the process $\{w_t\}$. In Kirman and Teyssière (2002b) and Teyssière (2003), we considered some tests for change–points motivated by the recent works of Kokoszka and Leipus (1999, 2000), Horváth, Kokoszka and Teyssière (2001) and Kokoszka and Teyssière (2002), who proposed some nonparametric and parametric tests for change–points in ARCH/GARCH processes. These tests were able to detect the presence of change–points in the volatility series generated by the processes presented in sub-sections 3.1 and 3.2.

Kokoszka and Teyssière (2002) and Teyssière (2003) resorted to a wavelet estimator of the scaling parameter for adjudicating between LRD and change–point processes. Unlike Fourier analysis, upon which is based the estimation of the scaling parameter in the frequency domain, wavelet analysis is a multi–resolution analysis. Then the wavelet estimation of the scaling parameter will be unaffected by changes in regime; interested readers are referred to Abry, Flandrin, Taqqu and Veitch (2000, 2003) for further details on wavelet analysis. By comparing the estimates of a semiparametric Gaussian estimator of long–range dependence and the wavelet based estimator, we can conclude whether the empirical LRD properties of asset price volatilities are either genuine or a statistical artefact.

We consider here two semiparametric estimators of the scaling parameter ϑ, the local Whittle estimator developed by Robinson (1995), and the wavelet based estimator proposed by Veitch and Abry (1999). Both estimators are based on the assumption that in a close positive neighborhood of the zero frequency, the spectrum of a long–memory process $\{Y_t\}$ has the form

$$f(\lambda) \approx c_f \lambda^{-\vartheta}, \quad \lambda \to 0_+, \tag{33}$$

where c_f is a constant. Robinson (1995) considered the local Whittle estimator and replaced in this estimator the expression of the spectrum by the approximation (33), and after concentrating in c_f, obtained the following estimator for the scaling parameter ϑ

$$\hat{\vartheta} = \arg\min_{\vartheta} \left\{ \ln\left(\frac{1}{m} \sum_{j=1}^{m} \frac{I_Y(\lambda_j)}{\lambda_j^{-\vartheta}} \right) - \frac{\vartheta}{m} \sum_{j=1}^{m} \ln(\lambda_j) \right\}, \tag{34}$$

where $I_Y(\lambda_j)$ is the periodogram evaluated on a set of m Fourier frequencies $\lambda_j = 2\pi j/T, j = 1, \ldots, m \ll [T/2]$, where $[\cdot]$ denotes integer part, the bandwidth parameter m tends to infinity with the sample size T but more slowly since $1/m + m/T \to 0$ as $T \to \infty$. Under appropriate conditions, which include the differentiability of the spectrum near the zero frequency and the existence of a moving average representation, the estimator has the following asymptotic distribution

$$\sqrt{m}(\hat{\vartheta} - \vartheta) \xrightarrow{d} N(0, 1), \tag{35}$$

where $\xrightarrow{d}$ means convergence in distribution. We select m by using the automatic bandwidth procedure proposed by Henry and Robinson (1996) and studied further by Henry (2001, 2003).

Veitch and Abry (1999) proposed an estimator of ϑ, which uses the independence properties of the wavelet coefficients $d_Y(j,k)$ for fractional Gaussian noise and related LRD processes; see Flandrin (1992). The wavelets coefficients are defined as

$$d_Y(j,k) = \langle Y, \psi_{j,k} \rangle, \tag{36}$$

where $\psi_{j,k}$ is a family of wavelet basis functions $\left\{\psi_{j,k} = 2^{-j/2}\psi_0(2^{-j}t - k)\right\}$, $j = 1, \ldots, J$ are the octaves or scales, $k \in \mathcal{Z}$, ψ_0 is the mother wavelet, which has $\mathcal{N}$ zero moments with $\mathcal{N} \geq 1$, i.e.,

$$\int t^k \psi_0(t) dt \equiv 0, \quad k = 0, \ldots, \mathcal{N} - 1. \tag{37}$$

The number $\mathcal{N}$ is chosen by the user. By construction, the family of wavelet basis functions is scale invariant, which can be written as

$$E d_Y(j, \cdot)^2 = 2^{j\vartheta} c_f C, \quad \text{with} \quad C = \int |\lambda|^{-\vartheta} |\Psi_0(\lambda)|^2 d\lambda, \tag{38}$$

where $\Psi_0(\lambda)$ is the Fourier transform of the mother wavelet ψ_0. The scaling parameter ϑ is estimated from the slope of the linear regression

$$\log_2 \left(E d_Y(j, \cdot)^2 \right) = j\vartheta + \log_2(c_f C). \tag{39}$$

The method by Veitch and Abry (1999) allows one to jointly estimate both parameters (ϑ, c_f). We use this estimator with the Daubechies wavelets, with $\mathcal{N} = 2$. The wavelet estimator has approximately the following asymptotic distribution

$$\sqrt{T}(\hat{\vartheta} - \vartheta) \approx N\left(0, \frac{1}{\ln^2(2) 2^{1-j_1}}\right), \tag{40}$$

where j_1 is the lowest octave, the LRD behavior being captured by the octaves larger than j_1. The choice of j_1, i.e., the cutoff between short–range dependence and long–range dependence, is similar to the bandwidth selection problem in semiparametric/nonparametric statistics. Bardet *et al.* (2000) established the exact asymptotic distribution for this estimator, as the assumption of independence of the wavelets coefficients $d_Y(j,k)$ is not true for all LRD processes. Unfortunately, the result by Bardet *et al.* (2000) is not of practical use for statistical inference as the asymptotic variance for this estimator depends on the scaling parameter ϑ.

Teyssière (2003) and Kokoszka and Teyssière (2002) considered series of volatilities and co–volatilities of FX rates and stock prices and measured the degree of long–range dependence for these series with these two semiparametric estimators. They reported interesting empirical results on the discrepancy of the estimated intensity of strong dependence, as the estimated scaling parameter is far lower when measured with the wavelet estimator than when measured with the local Whittle estimator. This lead Kokoszka and Teyssière (2002) and Teyssière (2003) to consider that the occurrence of long–range dependence in volatility, e.g., of (F)IGARCH processes might be a large sample artefact of a non–homogeneous data generating process and to conduct a change–point analysis of these series. We pursue in this direction here and conduct a change–point analysis of the different volatility processes generated by our models. Interested readers are referred to the revised version of Kokoszka and Teyssière (2002) where these tests and other ones are discussed.

5 Simulations and testing

In Kirman and Teyssière (2002a, 2002b) and Teyssière (2003), we show that these models are able to generate the sort of strong dependence observed in the volatility of asset returns, and that the degree of long–memory is controlled by the swings in opinions, i.e., the parameters governing the herding behaviour of the process $\{w_t\}$. These parameters are tuned so that the simulated series of returns R_t are $I(0)$, and the degree of long–memory in the series $|R_t|^n$ generated by the microeconomic models are similar to the ones empirically observed. We use the statistics proposed by Lo (1991), Kwiatkowski *et al.* (1992) and Giraitis *et al.* (2003a, 2003b) for testing the memory properties of the simulated series, and evaluate the degree of long-memory in the volatility series by using the semiparametric estimators proposed by Robinson (1995) and Giraitis, Kokoszka, Leipus and Teyssière (2000).

5.1 Testing for bubbles

In the standard methodology for bubbles testing, the logarithms of asset prices is regressed on the logarithms of fundamentals,

$$\ln P_t = \rho_0 + \rho_1 \ln \bar{P}_t + \varepsilon_t. \tag{41}$$

as it is assumed that if asset prices P_t contain a bubble, then the sequence of residuals $\{\hat{\varepsilon}_t\}$ is not stationary. Tests for the existence of unit roots are then applied to the series of residuals $\hat{\varepsilon}_t$ of the regression

$$\hat{\varepsilon}_t = \theta \hat{\varepsilon}_{t-1} + e_t, \tag{42}$$

i.e., we test $H_0 : \theta = 1$ against $H_A : |\theta| < 1$, bubbles testing amounts to unit root testing. There is a huge literature on this issue of unit root testing, see e.g., Dickey and Fuller (1979), Phillips and Perron (1988), MacKinnon (1991, 1996), Zarepour and Knight (1991), Paparoditis and Politis (2003), Horváth and Kokoszka (2003), Jach and Kokoszka (2004) and Kokoszka and Parfionovas (2004), among others.

We consider here the bootstrap unit root tests by Paparoditis and Politis (2003) and Kokoszka and Parfionovas (2004), and the subsampling unit root tests by Jach and Kokoszka (2004). These methods are based on the centered residuals $\bar{\hat{e}}_t = \hat{e}_t - (T-1)^{-1} \sum_{j=2}^{T} \hat{e}_j$, $t = 2, \dots, T$ from the regression (42), and consider bootstrap or subsampling data generating processes which satisfy the null hypothesis H_0 for obtaining a better approximation of the null distribution of the test statistics, and then increase the power of these tests.

The residual block bootstrap method by Paparoditis and Politis (2003) consists in constructing B block–bootstrap series satisfying H_0 as follows: for each block bootstrap sample k, $k = 1, \dots, B$, we choose an integer $b = b(T) < T$ such that $1/b + b/\sqrt{T} \to 0$ as $T \to \infty$, define $\kappa = [(T-1)/b]$, draw with replacement κ integers $i_0, \dots, i_{\kappa-1}$ from the set $\{1, \dots, T-b\}$, and build the bootstrap samples $\varepsilon_j(k)$ as follows

$$\begin{aligned} \varepsilon_1(k) &= \hat{\varepsilon}_1, \quad \varepsilon_j(k) = \varepsilon_{j-1}(k) + \bar{\hat{e}}_{i_m+s}, \quad j = 2, \dots, l, \quad l = \kappa b + 1, \\ m &= [(j-2)/b)], \quad s = j - mb - 1, \quad k = 1, \dots, B. \end{aligned} \tag{43}$$

Let $\hat{\theta}_T$ be the least squares (LS) estimator of θ in equation (42) and $\tilde{\theta}_l(k)$ be the LS estimator of the regression of $\varepsilon_j(k)$ on $\varepsilon_{j-1}(k)$. For a test of size γ, we reject H_0 if $T(\hat{\theta}_T - 1) < q_{l,B}(\gamma)$ where $q_{l,B}(\gamma)$ is the γ^{th} quantile of the distribution of $l(\tilde{\theta}_l(\cdot) - 1)$.

The tests by Horváth and Kokoszka (2003), Kokoszka and Parfionovas (2004), and Jach and Kokoszka (2004) take into account that in the standard unit root regression (42), the process $\{e_t\}$ might have heavy tails, i.e., $\Pr(|e_t| > x) \sim x^{-\alpha}$ as $x \to \infty$ for some $\alpha > 0$, α is called the tail index. This assumption is relevant in our case since some financial time series might not have a finite variance, i.e., $\alpha < 2$, and indeed some of our generated series exhibit a large variance. For $\alpha \geq 2$ the standard asymptotic theory for unit root tests is valid.

Horváth and Kokoszka (2003) and Jach and Kokoszka (2004) made the mild hypothesis that the $E(e_t) = 0$ and that the $\{e_t\}$ are in the domain of attraction of an α–stable law with $\alpha \in (1, 2)$, while Kokoszka and Parfionovas (2004) considered the more general case $\alpha \in (1, 2]$ which then includes the Gaussian case $\alpha = 2$. Chan and Tran (1989) have shown that under the null hypothesis H_0

$$T(\hat{\theta}_T - 1) \xrightarrow{d} \xi := \frac{\int_0^1 L_\alpha(\tau-) dL_\alpha(\tau)}{\int_0^1 L_\alpha^2(\tau) d\tau}, \tag{44}$$

where $\hat{\theta}_T$ is the LS estimator of θ in equation (42), $\{L_\alpha(\tau), \tau \in [0, 1)\}$ is an α–stable Levy process and $L_\alpha(\tau-)$ denotes the left limit of $L_\alpha(\cdot)$ at τ; see chapter 15 of Rachev and Mittnik (2000) for further details on this process. The purpose of the unit root and subsampling tests considered here is to approximate the distribution of the unit root statistic ξ without knowledge of the tail index α which is difficult to estimate; see Embrechts *et al.* (1997).

Horváth and Kokoszka (2003) demonstrated that the distribution of the unit root statistic ξ can be approximated using residual bootstrap, where the bootstrap sample size m satisfies the bandwidth condition for processes with infinite variance $m/T + 1/m \to 0$ as $T \to \infty$ insuring the convergence of the bootstrap distribution $m(\tilde{\theta}_m - \hat{\theta}_T)$ to ξ.

Kokoszka and Parfionovas (2004) construct B bootstrap samples satisfying H_0 as

$$\varepsilon_1(k) = \hat{\varepsilon}_1, \quad \varepsilon_j(k) = \varepsilon_{j-1}(k) + e_j(k), \quad j = 2, \ldots, m, \quad k = 1, \ldots, B, \tag{45}$$

where the $e_j(k)$ are randomly drawn with replacement from the $\bar{\hat{e}}_t$. Denote by $\tilde{\theta}_m(k)$ the LS estimator of the regression of $\varepsilon_j(k)$ on $\varepsilon_{j-1}(k)$, the distribution of $T(\hat{\theta}_T - 1)$ is approximated by the distribution of $m(\tilde{\theta}_m(\cdot) - 1)$. The choice of m is still an open question, although results reported in Kokoszka and Parfionovas (2004) show that taking $m = T$ yields a precise test. We choose $m = [0.9T]$.

The subsampling method advocated by Jach and Kokoszka (2004) consists in constructing $T - b$ processes which satisfies H_0, i.e.,

$$\varepsilon_1(k) = \bar{\hat{e}}_k, \ldots, \quad \varepsilon_b(k) = \bar{\hat{e}}_k + \ldots + \bar{\hat{e}}_{k+b-1}, \quad k = 2, \ldots, T - b + 1, \tag{46}$$

where b is the size of the subsampling blocks. Let $\tilde{\theta}_b(k)$ be the LS estimator of the regression of $\varepsilon_j(k)$ on $\varepsilon_{j-1}(k)$, we estimate the distribution of ξ by the one of $b(\tilde{\theta}_b(\cdot) - 1)$. As in Jach and Kokoszka (2004), we set $b = [0.15T]$ and use their correction factor for the critical region of the test.

After Evans (1991) who pointed out that unit root based procedures were unable to detect a class of bubbles, Wu and Xiao (2002) have shown that unit root tests do not detect collapsible bubbles and have proposed a procedure, close in spirit to the cointegration tests by Xiao and Phillips (2002), and based on the magnitude of variation of the partial sum processes $S_k = \sum_{t=1}^k \hat{\varepsilon}_t$ of the residuals of regression (41). If there is no bubble, the magnitude of fluctuation of the process $\{S_k\}$ is proportional to $k^{1/2}$, while the presence of a bubble makes the process $\{S_k\}$ diverging to ∞.

The statistic proposed by Wu and Xiao (2002) is based on the partial sum processes S_k^+ of the transformed residuals $\tilde{\varepsilon}_t^+$, so as to obtain a statistic, the limiting distribution of which under the null hypothesis is unaffected by any consequences of the serial correlation and the correlation between the residuals $\hat{\varepsilon}_t$ and the fundamentals $\bar{P}_t$. Under the null hypothesis of no bubbles this statistic, denoted by $\underline{R}$, converges to the supremum of a rather complex functional of Brownian motions, i.e.,

$$\underline{R} := \max_{1\leq k\leq T} \frac{k}{\hat{\omega}_{\varepsilon.d}\sqrt{T}} \left|k^{-1}S_k^+ - T^{-1}S_T^+\right| \xrightarrow{d} \sup_{0\leq\tau\leq 1} \left|\tilde{V}(\tau)\right|, \tag{47}$$

where $\tilde{V}(\tau) = \underline{W}_d(\tau) - \tau\underline{W}_d(1)$, $\underline{W}_d(\tau) = W_1(\tau) - \left[\int_0^1 dW_1 S'\right]\left[\int_0^1 SS'\right]^{-1}\int_0^\tau S$, $S(\tau)' = (1, W_2(\tau))$, $W_1(\tau)$ and $W_2(\tau)$ are Brownian motions that are independent of each other, $\hat{\omega}_{\varepsilon.d}$ is a nonparametric long–run variance estimator. For the estimation of $\hat{\omega}_{\varepsilon.d}$ we use the Bartlett lag window with the bandwidths used by Wu and Xiao (2002), i.e., $M_1 = [4m^\star]$, $M_2 = [6m^\star]$ and $M_3 = [8m^\star]$, with $m^\star = (m/100)^{1/4}$ and $m = [T^{0.9}]$. We tabulate the statistic $\underline{R}$ for the sample size $T = 1500$ used in our simulations using the quantiles of 100,000 simulations of this statistic for a process where the price P_t differs from the fundamentals $\bar{P}_t$ by a white noise random variable, the fundamentals following a random walk, see equation (1). We compare the results of this procedure with the ones provided by standard unit root tests which serve as benchmarks.

5.2 Testing for changes in the volatility

5.2.1 Parametric and semiparametric tests

We consider a first class of tests for detecting change–points in the volatility, which are based on the assumption that there is a unique change–point in the volatility process, which is assumed to follow an ARCH/GARCH process. The parameters of the ARCH/GARCH process are not constant and are changing at an unknown time denoted by t_0, e.g., for an ARCH(∞) process:

$$\begin{aligned} R_t &= m_R + \varepsilon_t, \quad E(\varepsilon_t) = 0, \quad \mathrm{Var}(\varepsilon_t|I_t) = \sigma_t^2, \\ \sigma_t^2 &= \omega + \sum_{j=1}^{\infty} \alpha_j \varepsilon_{t-j}^2, \quad t = 1, \ldots, t_0, \\ \sigma_t^2 &= \omega^\star + \sum_{j=1}^{\infty} \alpha_j^\star \varepsilon_{t-j}^2, \quad t = t_0+1, \ldots, T. \end{aligned} \tag{48}$$

Under the null hypothesis $H_0 : \omega = \omega^\star,\ \alpha_j = \alpha_j^\star$ for all j, while under the alternative hypothesis $H_A : \omega \neq \omega^\star$ or $\alpha_j \neq \alpha_j^\star$ for some j. Kirman and Teyssière (2002b) and Teyssière (2003) used the CUSUM based estimator for change-point in ARCH(∞) processes proposed by Kokoszka and Leipus (2000). One of the assumptions for this estimator is that the unconditional variance of the process changes after t_0, which is relevant for our purpose as the intensity of long–memory in the volatility spuriously generated by a non–homogeneous ARCH/GARCH type processes is linked to the magnitude in the change of the unconditional variance. This CUSUM estimator of the change–point time is based on the process $\{U_T(\tau), \tau \in [0,1]\}$

$$U_T(\tau) := \sqrt{T}\frac{[T\tau](T-[T\tau])}{T^2}\left(\frac{1}{[T\tau]}\sum_{j=1}^{[T\tau]} R_j^2 - \frac{1}{T-[T\tau]}\sum_{j=[T\tau]+1}^{T} R_j^2\right), \tag{49}$$

and is defined by

$$\hat{t} = [T\hat{\tau}], \quad \hat{\tau} = \min\left\{\tau : |U_T(\tau)| = \max_{0<\tau\leq 1} |U_T(\tau)|\right\}. \tag{50}$$

Kokoszka and Leipus (2000) applied this test to two series of returns on FX rates, the German Mark–US dollar and the British Pound–US dollar FX rates, and detected a change in regime in September 1979 which might be the consequence of the inception of the European Monetary System in April 1979. Teyssière (1997) modeled the conditional variance and covariance of the same series by a bivariate long–memory ARCH process, and found a change in the long–memory structure of the conditional covariance matrix after that inception date. These two results illustrate the interplay of strong dependence and change–point for volatility processes.

We applied this CUSUM test to the series R_t generated by the microeconomic models. A graphical representation of the process $\{w_t\}$ shows that this test detects changes in the heteroskedastic structure which matches the switches in the process $\{w_t\}$. An illustration of the performance of this test is provided by Figures 1 and 2, other examples are reported in Kirman and Teyssière (2002b). We detect here the occurrence of change–point with the test developed by Kokoszka and Leipus (1999), that is based on the process $\{U_T(\tau), \tau \in [0,1]\}$, which under the null hypothesis of no change–point converges to the process $\left\{\sigma W^0(\tau), \tau \in [0,1]\right\}$, i.e.,

$$U_T(\tau) \xrightarrow{\mathcal{D}[0,1]} \sigma W^0(\tau), \tag{51}$$

where $\xrightarrow{\mathcal{D}[0,1]}$ means weak convergence in the space $\mathcal{D}[0,1]$ endowed with the Skorokhod topology, $W^0(\tau)$ is the Brownian bridge on the unit interval $[0,1]$ defined as $W^0(\tau) = W(\tau) - \tau W(1)$, $W(\tau)$ is the Wiener process. We consider here as test statistic the functional based on the process $\{U_T(\tau), \tau \in [0,1]\}$

$$\sup_{0\leq\tau\leq 1} |U_T(\tau)|/\sigma \xrightarrow{d} \sup_{0\leq\tau\leq 1} \left|W^0(\tau)\right|, \tag{52}$$

where the long–run variance σ^2 is usually estimated by nonparametric kernel methods. We use here the heteroskedastic and autocorrelation consistent estimator by Newey and West (1987) with the truncations order $q = 0, 2, 5, 10, 15$, and the VARHAC estimator by Den Haan and Levin (1997), the order of dependence of which is selected with the Bayes information parsimony criteria.

We also consider parametric tests for change-point based on the empirical process of squared residuals of ARCH sequences proposed by Horváth, Kokoszka and Teyssière (2001), and further extended by Berkes and Horváth (2003) to the general GARCH(p,q) case. In Kokoszka and Teyssière (2002), we have seen that these tests have some power against non–homogeneous GARCH(1,1) sequences as well, provided that the unconditional variance of the process changes after t_0. We assume that for $t \leq t_0$, the observations follow the model

$$R_t = m_R + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_t^2), \quad \sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha\varepsilon_{t-1}^2, \tag{53}$$

the unconditional variance of the process is $\omega/(1-\alpha-\beta)$, while for $t > t_0$ the observations follow the model

$$R_t = m_R + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_t^2), \quad \sigma_t^2 = \omega^\star + \beta^\star\sigma_{t-1}^2 + \alpha^\star\varepsilon_{t-1}^2. \tag{54}$$

Under the null hypothesis of no change in the data, the observations R_t, $t = 1, \dots T$, follow the model (53), while under the alternative hypothesis the parameter vector (ω, β, α) changes at some unknown time t_0 so that $\omega/(1-\alpha-\beta) \neq \omega^\star/(1-\alpha^\star-\beta^\star)$.

We fit a GARCH(1,1) process on the sequence of returns $\{R_t\}$ generated by the microeconomic models, the tests being based on the sequential empirical process

$$\hat{K}_T(\tau, t) := T^{-1/2} \sum_{1 \le i \le [T\tau]} \left(\mathbf{1}(\hat{\varepsilon}_i^2 \le t) - F(t)\right), \quad 0 < \tau \le 1, \tag{55}$$

where $\mathbf{1}(\cdot)$ denotes the indicator function, $\hat{\varepsilon}_i^2$ are the squared residuals and $F(\cdot)$ is the density of the squared error terms ε_t^2. Kokoszka and Teyssière (2002) considered the Cramér–von Mises (CVM) type statistics as the CVM asymptotic test has the correct size, even for GARCH(1,1) processes without fourth moments. For $1 \le k \le T$ define

$$\hat{\mathcal{K}}(k, t) := \sqrt{T}\, \frac{k}{T}\left(1 - \frac{k}{T}\right)\left|\hat{F}_k(t) - \hat{F}_k^\star(t)\right|, \tag{56}$$

where

$$\hat{F}_k(t) := \frac{1}{k}\#\{i \le k : \hat{\varepsilon}_i^2 \le t\}, \quad \hat{F}_k^\star(t) := \frac{1}{T-k}\#\{i > k : \hat{\varepsilon}_i^2 \le t\}. \tag{57}$$

The CVM test consists in comparing the empirical distribution of the residuals before and after t_k. From the results in Horváth *et al.* (2001), for large T the Cramér − von Mises statistic has approximately the following asymptotic distribution

$$\hat{B} := \int_0^1 \left\{\frac{1}{T}\sum_{i=1}^{T}[\hat{\mathcal{K}}([T\tau], \hat{\varepsilon}_i^2)]^2\right\} d\tau \approx B := \int_0^1 \int_0^1 \mathcal{K}^2(\tau, u)\, du\, d\tau, \tag{58}$$

where $\{\mathcal{K}(\tau, u), 0 \le \tau, u \le 1\}$ is the tied–down Kiefer process, the square–integral of which has been studied in Blum, Kiefer and Rosenblatt (1961). The critical values for B, derived from Blum *et al.* (1961), are given in Kokoszka and Teyssière (2002).

Kokoszka and Teyssière (2002) introduced two tests for change–point in a GARCH(1,1) process, based on the generalized likelihood–ratio (GLR) principle. While the GLR tests still have some power for detecting changes in the parameters when the unconditional variance of the GARCH(1,1) process remains constant after the change–point time t_0, the CVM is more powerful for detecting single and multiple changes in the distribution of the innovations; see also Horváth *et al.* (2004). Furthermore, for GARCH(1,1) processes without fourth moments, correct inference for the GLR tests is provided by bootstrap based inference, which would be rather computing intensive for the sample size considered in this work.

5.2.2 Online detection of change–points

So far, we consider posterior tests for change–points, i.e., tests which are applied once the whole series $R_1, \dots, R_T$ has been generated and observed. The detection procedure is off–line. In contrast, online or sequential change–points tests are applied to the currently observed series for testing whether a change in the parameters recently occurred.

Mikosch and Stărică (1999) proposed a goodness–of–fit test for detecting changes in the parameters of a GARCH(1,1) sequence. If we assume that the sequence of returns $\{R_t\}$ follows a GARCH(1,1) process, the fourth moments of which do exist, under the null hypothesis of constancy of the parameters of this process into the interval $[R_1, R_m]$, we have

$$MS_m := \sqrt{m} \sup_{\lambda \in [0,\pi]} \left| \sum_{h=1}^{m-1} \frac{\gamma_m(h)}{[\mathrm{Var}(R_0 R_h)]^{1/2}} \frac{\sin(\lambda h)}{h} \right| \xrightarrow{d} \sup_{\lambda \in [0,\pi]} \left| W^0(\lambda) \right|, \tag{59}$$

where $W^0(\lambda)$ is a Brownian bridge on $[0, \pi]$, m is the window width of the so called "interval of homogeneity" $[R_1, R_m]$, $\gamma_m(h)$ are the sample autocovariances at lag h, the normalising factor $\mathrm{Var}(R_0 R_h)$ depends on the estimated parameters of the GARCH(1,1) model and the estimated fourth moments of the innovations; see Mikosch (1999) for further details. We consider a first interval of homogeneity $m = 125$ that we increment by a step–size equal to 1 until the null hypothesis of no change–point is rejected. Then, we resume the procedure from the detected change–point until the next detected change point or the end of the process.

5.2.3 Nonparametric tests for multiple change–points

These off–line tests do not make any assumption on the functional form of the volatility process, but assume that the process $\{R_t\}$ is piecewise stationary. After a suggestion by Murad Taqqu, we consider tests for change in the unconditional variance as Mikosch and Stărică (1999, 2004) claimed that the degree of long–range dependence in asset prices returns is the consequence of a change in the parameters of the conditional variance function, the intensity of strong dependence being proportional to the magnitude of the change in the unconditional variance of the process resulting from the changes in the parameters. Granger and Hyung (1999) considered the test for change in the variance of a process $\{R_t\}$ proposed by Inclán and Tiao (1994), which is based on the process $\{D_T(\tau), \tau \in [0,1]\}$ defined as

$$D_T(\tau) := \frac{\sum_{j=1}^{[T\tau]} R_j^2}{\sum_{j=1}^{T} R_j^2} - \frac{[T\tau]}{T}, \quad \tau \in [0,1]. \tag{60}$$

Under the null hypothesis of constant unconditional variance, the process $\{D_T(\tau), \tau \in [0,1]\}$ converges to a Brownian bridge on $[0,1]$. A test for constancy of the unconditional variance is based on the following functional of the process $\{D_T(\tau)\}$, which under this null hypothesis of constant unconditional variance converges in distribution to the supremum of a Brownian bridge on $[0,1]$

$$\sqrt{T/2} \sup_{0 \le \tau \le 1} |D_T(\tau)| \xrightarrow{d} \sup_{0 \le \tau \le 1} \left| W^0(\tau) \right|. \tag{61}$$

Given that we are also considering large samples, i.e., over 1000 observations, the occurrence of a unique change point is unlikely. We then have to find the configuration $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots, \tau_K)$, i.e., the set of break fractions $\{\tau_0 = 0 < \tau_1 < \ldots < \tau_{K-1} < \tau_K = 1\}$ so that the $K - 1$ change–points occur at times $t_j = [T\tau_j]$, $j = 1, \ldots, K-1$. The binary segmentation procedure is the standard method for detecting multiple change–points with a test for single change–point, by splitting the series at a detected change–point and repeat the detection procedure on the new segments until no further change–point is found. Vostrikova (1981) studied and proved the consistency of this method; further references on this issue are given in Brodsky and Darkhovsky (1993). We use this binary segmentation procedure with the tests by Inclán and Tiao (1994) and Kokoszka and Leipus (1999) for detecting multiple changes in the variance of the returns generated by the microeconomic models.

Specific tests for multiple change–points have been considered by Lavielle (1999), Lavielle and Moulines (2000), and other authors cited as reference in these papers. One approach consists in detecting the changes in the mean of the volatility proxy series, i.e., the absolute returns $|R_t|$ and squared returns R_t^2 produced by the microeconomic models. However, since the shifts in the mean of the series $|R_t|$ and R_t^2 are the direct consequences of the changes in the unconditional variance of the level series R_t, the straight approach consists in detecting these changes in the unconditional variance.

We then consider the test for multiple changes in variance, which is a particular case of Lavielle's (1999) test for multiple changes in distribution, i.e., changes in mean and variance. An interesting property of this method is that the rate of convergence to the true configuration $\boldsymbol{\tau}^\star = (\tau_0^\star, \tau_1^\star, \ldots, \tau_K^\star)$ does not depend on the dependence structure of the data.

Let $\theta = (\sigma_1^2, \ldots, \sigma_K^2) \in \Theta$ be the parameters of the process, and $\mathcal{T}$ be the set of configurations $\boldsymbol{\tau}$. We estimate the configuration of the series $\boldsymbol{\tau}$, the number of change points, and the vector of parameters θ, by minimizing the penalized contrast function, i.e.,

$$(\hat{\boldsymbol{\tau}}, \hat{\theta}, \hat{K}) = \arg \min_{1 \le K \le \bar{K}} \inf_{(\tau,\theta)\in(\mathcal{T}\times\Theta)} \left\{ \frac{1}{T} \sum_{j=1}^{K} \left(\frac{\|\boldsymbol{R}_j - \bar{m}_R\|^2}{\sigma_j^2} + n_j \ln \sigma_j^2 \right) + \beta_T K \right\}, \tag{62}$$

where $\bar{K}$ is the upper bound for the dimension, $\bar{m}_R$ is the empirical mean of the returns process R_t, $\boldsymbol{R}_j$ denotes the set of observations R_t which belong to the j^{th} segment, and n_j is the number of observations in that segment. The penalty term is $\beta_T K$, where the sequence $\{\beta_T\}$ satisfies $\beta_T \to 0$ and $T^{2-h}\beta_T \to \infty$ as $T \to \infty$, with $h \in [1, 2)$. The parameter β_T controls the resolution level of the segmentation $\tilde{\boldsymbol{R}} = \{\boldsymbol{R}_1, \ldots, \boldsymbol{R}_K\}$ and should be optimally chosen so that the dimension K is neither underestimated nor overestimated: if the resolution parameter β_T is too large only few breaks are detected, while if β_T is too small all small changes are detected; see Lavielle (1998) for further details. Since Lavielle (1999) underlines the fact that there is no automatic rule for the choice of β_T, we consider here the formulas suggested by Yao (1988), i.e., $\beta_T = \ln(T)/T$, and by Lavielle and Moulines (2000) for the case of strongly dependent processes, i.e., $\beta_T = 4\ln(T)/T^{1-\vartheta}$ where $\vartheta \in (0, 1)$ is the scaling parameter of the series that we estimate with the wavelet estimator defined in equation (39). The maximum dimension is set to $\bar{K} = 50$. We use here the iterative conditional mode algorithm, which is not as optimal as the simulated annealing algorithm as it can be "trapped" by local optima in the optimisation problem (62), but is computationally feasible for the large number of simulations considered here.

5.3 Simulation results

The details of the simulation are as follows:

- $T = 1500$, (sample size). We avoid transient effects by discarding for each simulated series the first 200 observations and by keeping the next T ones.
- number of simulated series = 10000,
- number of agents $N = 1000$,
- the number of fundamentalists at the beginning of the process = $N/2$, thus $k_0 = 0.5$,
- $P_0 = 1000$,

- $\bar{P}_0 = 1050$,
- annual domestic interest rate, $r = 0.04$, the daily domestic interest rate is 0.000133668,
- annual foreign interest rate, $\rho = 0.07$, the daily foreign interest rate is 0.00018538.
- For the forecast functions, $h_0 = 0.6250$, $h_1 = 0.3685$, $\nu_1 = 0.6000$,
- For the learning model, the memory of agents $M = 10$, while $\varkappa = 0.09$ and $\eta = 12$.

Notice that the choice of the parameters is not unique, as other configurations can produce similar results. The series of error terms have been generated by using the Box-Muller transformation, the sequence of uniform deviates used by this transformation succeeds Marsaglia's (1996) DIEHARD tests.

A series of standard tests were now run on the simulated data to see if it was possible to reject alternative specifications which have different statistical characteristics from those of the process used in the model. In particular it is interesting to see if an econometrician faced with this data would have been able to detect the bubbles in the level series and the changes in regimes in the volatility. Results are displayed on p 30 ff.

The presence of bubbles was quite always detected by the test by Wu and Xiao (2002) for all the choices of the bandwidth parameter. We observe that the power of this test monotonically decreases as the bandwidth increases. The standard methodology based on unit root testing was not able to detect the bubbles, a result which is casting doubts on the relevance of these methods for the detection of bubbles on large samples, which are likely to contain collapsible bubbles. Interestingly, the subsampling method by Jach and Kokoszka (2004) rejects less often the null hypothesis of unit root than the tests by Paparoditis and Politis (2003) and Kokoszka and Parfionovas (2004), and then detects 40 % of the time the presence of bubbles.

The wavelet analysis of the long range dependence parameter shows that the average estimated scaling parameter (0.1389) of the absolute returns is far below the one measured with the standard local Whittle estimator (0.6270). A similar result is observed for squared returns. This result, which is similar to what is empirically observed on real data, suggests the presence of changes in regimes in the series of absolute and squared returns.

Nonparametric tests for multiple change points and the parametric sequential test by Mikosch and Stărică (1999) for detecting changes in a GARCH(1,1) process detected several changes in the volatility process. The test by Kokoszka and Leipus (1999) and Inclán and Tiao (1994) detected several changes in regime, but less than the number of change-points detected by Lavielle's (1999) test. The result of the test by Kokoszka and Leipus (1999) is not too much affected by the choice for the estimator $\hat{\sigma}$. The results of the test by Lavielle (1999) strongly depend on the penalty term β_T. Table 2 reports the results with $\beta_T = 4\ln(T)/T^{1-\vartheta}$. For $\beta_T = \ln(T)/T$, i.e., with the Bayes criterion, the average number of detected change–points increases to 22.3804, while for $\beta_T = 4\ln(T)/T$ on average 2.2327 change–points are detected. Since the results for $\beta_T = 4\ln(T)/T^{1-\vartheta}$ are consistent with the results of the tests by Inclán and Tiao (1994) and Kokoszka and Leipus (1999), the choice of this resolution parameter looks sensible.

The goodness–of–fit test by Mikosch and Stărică (1999) rejects more frequently the null hypothesis of homogeneous conditional volatility. However, a rejection of the null hypothesis might be the consequence of the fact that the data are more volatile than the ones implied by a GARCH(1,1) process, which is rather likely in our case.

The hypothesis of no change in regime was rejected 30 % of the time by the CVM test. A limited number of replications with bootstrap based inference seems to indicate that bootstrap tests have a higher power.

Finally, the statistical properties of the generated series do not strongly differ between the opinion diffusion models presented here, which is not surprising since we tune the parameters of the microeconomic models so as to obtain statistical properties close to the ones of real financial time series.

6 Conclusion

In this paper we have discussed the problem of bubbles, their nature and in particular their detection in empirical time series. The nature of bubbles is somewhat ambiguous but can be summarised as a departure from fundamentals. We have presented several versions of a model which lead to such departures and where there is always a return to these fundamentals.

Our model suggests that the underlying reasons for the bubbles phenomena are that there are switches in expectations caused by individuals changing their forecasting rules. There is, in our model, a tendency for these changes to be self reinforcing.

This leads to regime shifts albeit not of the sort typically found in the literature. We discuss the econometric methods available to detect the change points in this process. Standard unit roots tests perform poorly on the series generated by our model. This is not surprising since the process is indeed a unit root one for some of the time. Nevertheless some of the recent tests that we use do remarkably well in picking up these shifts.

The regime shifts are important since it is often argued that the property of "long memory" in financial series is often spurious and due to switching regimes. Our analysis shows that there is a combination of "genuine" long memory in the physical sense and of the switching regime effect. To be more precise, a wavelet analysis of the series generated by our models shows that the strong persistence in the volatility is likely to be the outcome of a mix of changes in regimes and a moderate level of long–range dependence.

We seem thus to have made significant progress towards the detection of bubbles. At the same time there is the problem of the different possible origins of such phenomena.

Our model is but one of the possible explanations for the bubbles phenomenon, and others such as the presence of expectations of the expectations of others; see Allen *et al.* (2003). The tests proposed should be a step towards enabling us to distinguish between these explanations. Another interesting route is to try to differentiate between the sort of chaotic models proposed for example by Brock and Hommes (1997) and the stochastic process that we propose. Recent work by Bhansali, Holland and Kokoszka (2003a, 2003b) shows that chaotic intermittent dynamics can produce long memory and it would be an interesting research problem to see to what extent the distinction between this sort of dynamic and that of our simple stochastic process can be made.

References

Abry, P., Flandrin, P., Taqqu, M.S., Veitch, D., (2003). Self–similarity and long–range dependence through the wavelet lens, in: Doukhan, P., Oppenheim, G., Taqqu, M.S. (Eds.), Long–Range Dependence: Theory and Applications. Birkhauser, Boston, pp. 527–556.

Abry, P., Flandrin, P., Taqqu, M.S., Veitch, D., (2000). Wavelets for the analysis, estimation, and synthesis of scaling data, in: Park, K., Willinger, W. (Eds.), Self Similar Network Traffic and Performance Evaluation. Wiley, pp. 39–88.

Allen, F., Morris, S., Shin, H-S., (2003). Beauty contests, bubbles and iterated expectations in asset markets. Preprint.

Aoki, M. (1996). A New Approach to Macroeconomic Modeling. Cambridge University Press, New York.

Arthur, W.B., Ermoliev, Yu.M., Kaniovski, Yu.M., (1986). Strong laws for a class of path–dependent urn processes, in: Arkin, V.I., Shiryaev, A., Wets, R. (Eds.), Proceedings of International Conference on Stochastic Optimisation, Kiev 1984. Lecture Notes in Control and Information Sciences, Vol. 81. Springer Verlag, New York.

Avery, C., Zemsky, P., (1998). Multidimensional uncertainty and herd behavior in financial markets. American Economic Review 88, 724–748

Bacchetta, P., van Wincoop, E., (2003). Can information heterogeneity explain the exchange rate determination puzzle? Preprint.

Bardet, J.M., Lang, G., Moulines, E., Soulier, P., (2000). Wavelet estimator of long range dependent processes. Statistical Inference for Stochastic Processes 3, 85–99

Barrow, S., (1994). Fundamental currency forecasting: an agnostic view. Chemical Working Papers in Financial Economics 3, September, 8–12.

Basseville, M., Nikiforov, I.V., (1993). Detection of Abrupt Changes: Theory and Applications. Prentice Hall, Englewood Cliffs, NJ.

Beran, J., (1994). Statistics for Long–Memory Processes. Chapman & Hall, New York.

Berkes, I., Horváth, L., (2003). Limit results of the empirical process of squared residuals in GARCH models. Stochastic Processes and their Applications 105, 271–298.

Berkes, I., Horváth, L., Kokoszka, P.S., (2003). Testing for parameter constancy in GARCH(p, q) models. Statistics and Probability Letters, forthcoming.

Berkes, I., Gombay, E., Horváth, L., Kokoszka, P.S., (2004). Sequential change–point detection in GARCH(p, q) models. Econometric Theory 20, 1140–1167.

Bhansali, R.J., Holland, M.P., Kokoszka, P.S., (2003a). Chaotic maps with slowly decaying correlations and intermittency. Fields Institute Communications, **44,** 99–126.

Bhansali, R.J., Holland, M.P., Kokoszka, P.S., (2003b). Intermittency, long–memory and financial returns, in: Teyssière, G., Kirman, A. (Eds.), Long–Memory in Economics. Springer Verlag, Berlin, forthcoming.

Blanchard, O., Watson, M.W., (1982). Bubbles, rational expectations and financial markets, in: Wachtel, P. (Ed.), Crisis in the Economic and Financial System. Lexington, MA, Lexington Books, pp. 295–315.

Blum, J.R., Kiefer, J., Rosenblatt, M., (1961). Distribution free tests of independence based on the sample distribution function. The Annals of Mathematical Statistics 32, 485–498.

Brock, W.A., Hommes, C.H., (1999). Rational animal spirits, in: Herings, P.J.J., van der Laan, G., Talman, A.J.J. (Eds.), The Theory of Markets. North Holland, Amsterdam, pp. 109–137.

Brock, W.A., Hommes, C.H., (1997). A rational route to randomness. Econometrica 65, 1059–1095.

Brodsky. B., Darkhovsky, B., (1993). Nonparametric Methods in Change–Point Problems. Kluwer Academic Publishers, The Netherlands.

Caginalp, G., Porter, D., Smith, V.L., (1998). Experimental asset markets, in: Earl, P.E., Kemp, S. (Eds), The Elgar Companion to Consumer Research and Economic Psychology. Edwar Elgar, Cheltenham, UK, pp. 215–222.

Campbell, J.Y., Shiller, R.J., (1987). Cointegration and tests of present value models. Journal of Political Economy 95, 1062–1088.

Chan, N.H., Tran, L.T., (1989). On the first order autoregressive process with infinite variance. Econometric Theory 5, 354–362.

Cheung, Y-W., Chinn, M.D., Pascual, A.G., (2002) Empirical exchange rate models in the nineties: Are they fit to survive. NBER. Working Paper, 9393.

Chionis, D., MacDonald, R., (2002). Aggregate and disaggregate measures of the foreign exchange risk premium. International Review of Economics and Finance 11, 57–84.

Csörgő, M., Horváth, L., (1997). Limit Theorems in Change-Point Analysis. Wiley.

Den Haan, W.J., Levin, A., (1997). A practitioner's guide to robust covariance matrix estimation, in Maddala, G.S., Rao, C.R. (Eds.), Handbook of Statistics, Vol. 15. North Holland, Amsterdam, pp. 291–341.

De Long, J.B., Schleifer, A., Summers, L.H., Waldmann, R.J., (1989). The size and incidence of the losses from noise trading. Journal of Finance 44, 681–699.

Diba B.T., Grossman, H.I., (1988). Explosive rational bubbles in stock prices? American Economic Review 78, 520–530.

Dickey, D.A., Fuller, W.A., (1979). Distribution of the estimators for autoregressive time series with a unit root. Journal of the American Statistical Association 74, 427–431.

Ding, Z., Granger, C.W.J., (1996). Modelling volatility persistence of speculative returns. A new approach. Journal of Econometrics 73, 185–215.

Donaldson, R.G., Kamstra, M., (1996). A new dividend procedure that rejects bubbles in asset prices: The case of the 1929's stock crash. The Review of Financial Studies 9, 333–383.

Ellison, G., Fudenberg, D., (1993). Rules of thumb for social learning. Journal of Political Economy 101, 612–643.

Embrechts, P., Klüppelberg, C., Mikosch, T., (1997). Modelling Extremal Events for Insurance and Finance. Springer Verlag, Berlin.

Engle, R.F., (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflation. Econometrica 50, 987–1007.

Evans, G.W., (1991). Pitfalls in testing for explosive bubbles in asset prices. American Economic Review 81, 922–930.

Evans, G.W., (1986). A test for speculative bubbles in the Sterling Dollar exchange rate. American Economic Review 76, 621–636.

Flandrin, P., (1992). Wavelet analysis and synthesis of fractional Brownian motion. IEEE Transactions on Information Theory 38, 910–917.

Flood, R.P., Garber, P.M., (1980). Market fundamentals versus price level bubbles: the first tests. Journal of Political Economy 88, 745–770.

Flood, R.P., Hodrick, R.J., (1990). On testing for speculative bubbles. Journal of Economic Perspectives 4, 85–102.

Frankel, J., Rose, A.K., (1995). Empirical research on nominal exchange rates, in: Grossman, G.M., Rogoff, K. (Eds.), Handbook of International Economics, Vol. 3. North-Holland, Amsterdam, pp. 1680–1729.

Frankel, J.A., Froot, K.A., (1986). The dollar as an irrational speculative bubble: the tale of fundamentalists and chartists. The Marcus Wallenberg Papers on International Finance 1, 27–55.

Garber, P.M., (2000). Famous First Bubbles. MIT Press.

Giraitis, L., Leipus, R., Surgailis, D., (2003). Recent advances in ARCH modelling, in: Teyssière, G., Kirman, A. (Eds.), Long–Memory in Economics. Springer Verlag, Berlin, forthcoming.

Giraitis, L., Kokoszka, P.S., Leipus, R., Teyssière, G., (2003a). Rescaled variance and related tests for long–memory in volatility and levels. Journal of Econometrics 112, 265–294.

Giraitis, L., Kokoszka, P.S., Leipus, R., Teyssière, G., (2003b). On the power of R/S-type tests under contiguous and semi long–memory alternatives. Acta Applicandae Mathematicae 78, 285–299.

Giraitis, L., Kokoszka, P.S., Leipus, R., (2000). Stationary ARCH models: dependence structure and central limit theorem. Econometric Theory 16, 3–22.

Giraitis, L., Kokoszka, P.S., Leipus, R., Teyssière, G., (2000). Semiparametric estimation of the intensity of long–memory in conditional heteroskedasticity. Statistical Inference for Stochastic Processes 3, 113–128.

Granger, C.W.J., (2002). Long memory, volatility, risk and distribution. Preprint.

Granger, C.W.J., Hyung, N., (1999). Occasional structural breaks and long–memory. Preprint.

Granger, C.W.J., Ding, Z., (1995). Some properties of absolute returns, an alternative measure of risk. Annales d'Économie et de Statistique 40, 67–91.

Haken, H., (1977). Synergetics. Springer Verlag, Berlin.

Hamilton, J.D., Whiteman, C.H., (1985). The observable implications of self–fulfilling expectations. Journal of Monetary Economics 16, 353–373.

Henry, M., Robinson, P.M., (1996). Bandwidth choice in Gaussian semiparametric estimation of long range dependence, in: Robinson, P.M., Rosenblatt, M. (Eds.), Athens Conference on Applied Probability and Time Series Analysis, Time Series Analysis, In Memory of E.J. Hannan. Lecture Notes in Statistics, Vol. 115. Springer Verlag, New York, pp. 220–232.

Henry, M., (2003). Bandwidth choice, optimal rates and adaptivity in semiparametric estimation of long memory, in: Teyssière, G., Kirman, A. (Eds.), Long–Memory in Economics. Springer Verlag, Berlin, forthcoming.

Henry, M., (2001). Robust automatic bandwidth for long–memory. Journal of Time Series Analysis 22, 293–316.

Horváth, L., Kokoszka, P.S., (2003). A bootstrap approximation to a unit root test statistic for heavy tailed observations. Statistics and Probability Letters 62, 163–173.

Horváth, L., Kokoszka, P.S., Teyssière, G., (2004). Bootstrap misspecification tests for ARCH based on the empirical process of squared residuals. Journal of Statistical Computation and Simulation 74, 469–485.

Horváth, L., Kokoszka, P.S., Teyssière, G., (2001). Empirical process of the squared residuals of an ARCH sequence. The Annals of Statistics 29, 445–469.

Inclán, C., Tiao, G.C., (1994). Use of cumulative sums of squares for retrospective detection of changes in variance. Journal of the American Statistical Association 89, 913–923.

Jach, A., Kokoszka, P.S., (2004). Subsampling unit root tests for heavy–tailed observations. Methodology and Computing in Applied Probability 6, 73–97.

Kazakevičius, V., Leipus, R., (2003). A new theorem on existence of invariant distributions with applications to ARCH processes. Journal of Applied Probability 40, 147–162.

Kazakevičius, V., Leipus, R., (2002). On stationarity in the ARCH(∞) model. Econometric Theory 18, 1–16.

Kirman, A., (1993). Ants, rationality and recruitment. The Quarterly Journal of Economics 108, 137–156.

Kirman, A., (1991). Epidemics of opinion and speculative bubbles in financial markets, in: Taylor, M. (Ed.), Money and Financial Markets. Macmillan London, pp. 354–368.

Kirman, A., Teyssière, G., (2002a). Microeconomic models for long–memory in the volatility of financial time series. Studies in Nonlinear Dynamics and Econometrics 5, 281–302.

Kirman, A., Teyssière, G., (2002b). Bubbles and long–range dependence in asset prices volatilities, in: Hommes, C.H., Ramer, R., Withagen, C. (Eds.), Equilibrium, Markets and Dynamics. Essays in Honour of Claus Weddepohl. Springer Verlag, Berlin, pp. 307–327.

Kokoszka, P.S., Parfionovas, A., (2004). Bootstrap unit root tests for heavy tailed time series, in: Rachev, S.T. (Ed.), Handbook of Computational and Numerical Methods in Finance. Springer Verlag, pp. 175–197.

Kokoszka, P.S., Teyssière, G., (2002). Change–point detection in GARCH models: asymptotic and bootstrap tests. Under revision for the Journal of Business and Economic Statistics.

Kokoszka, P.S., Leipus, R., (2003). Detection and estimation of changes in regime, in: Doukhan, P., Oppenheim, G., Taqqu, M.S. (Eds.), Long–Range Dependence: Theory and Applications. Birkhauser, Boston, pp. 325–337.

Kokoszka, P.S., Leipus, R., (2000). Change–point estimation in ARCH models. Bernoulli 6, 513–539.

Kokoszka, P.S., Leipus, R., (1999). Testing for parameter changes in ARCH models. Lithuanian Mathematical Journal 39, 231–247.

Kouri, P.J.K., (1983). Balance of payments and the foreign exchange market: a dynamic partial equilibrium model, in: Bhandari, J.S., Putnam, B.H. (Eds.), Economic Interdependence and Flexible Exchange Rates. MIT Press, Cambridge, Massachussets.

Kwiatkowski, D., Phillips, P.C.B., Schmidt, P., Shin, Y., (1992). Testing the null hypothesis of stationarity against the alternative of a unit root: how sure are we that economic series have a unit root? Journal of Econometrics 54, 159–178.

Lavielle, M., (1999). Detection of multiple changes in a sequence of dependent variables. Stochastic Processes and their Applications 83, 79–102.

Lavielle, M., (1998). Optimal segmentation of random processes. IEEE Transactions on Signal Processing 46, 1365–1373.

Lavielle, M., Moulines, E., (2000). Least squares estimation of an unknown number of shifts in a time series. Journal of Time Series Analysis 21, 33–59.

LeBaron, B., (2001). Empirical regularities from interacting long- and short-memory investors in an agent-based stock market. IEEE Transactions on Evolutionary Computation 5, 442–455.

Le Roy, S.F., Porter, R.D., (1981). The present value relation: tests based on implied variance bounds. Econometrica 49, 555–574.

Lo, A.W., (1991). Long–term memory in stock market prices. Econometrica 59, 1279–1313.

Lyons, R., (2001). The Microstructure Approach to Exchange Rates. MIT Press, Cambridge, Massachussets.

MacKinnon, J.G., (1996). Numerical distribution functions for unit–root and cointegration tests. Journal of Applied Econometrics 11, 601–618.

MacKinnon, J.G., (1991). Critical values for cointegration tests, in: Granger, C.W.J., Mizon, G. (Eds.), Long Run Economic Relations: Readings in Cointegration. Oxford University Press, pp. 267–276.

Mankiw, N.G., Romer, D., Shapiro, M., (1985). An unbiased reexamination of stock market volatility. Journal of Finance 40, 677–689.

Marsaglia, G., (1996). DIEHARD: A Battery of Tests of Randomness. `http://stat.fsu.edu/pub/diehard`.

Marsh, T.A., Merten, R.C., (1986). Dividend variability and variance bounds tests for the rationality of stock market prices. American Economic Review 76, 483–498.

McCallum, B., (1983). On non–uniqueness in rational equilibrium models: An attempt at perspective. Journal of Monetary Economics 11, 139–168.

Meese, R.A., (1986). Testing for bubbles in exchange markets: the case of sparkling rates. Journal of Political Economy 94, 345–373.

Meese, R.A., Rogoff, K., (1983). Empirical exchange rate models of the seventies: do they fit out–of–sample? Journal of International Economics 14, 3–24.

Mikosch, T., (1999). Modelling financial time series. Preprint.

Mikosch, T., Stărică, C., (2004). Non–stationarities in financial time series: the long range dependence and the IGARCH effects. Review of Economics and Statistics, **86,** 278–290.

Mikosch, T., Stărică, C., (2000). Limit theory for the sample autocorrelations and extremes of a GARCH(1,1) process. The Annals of Statistics 28, 1427–1451.

Mikosch, T., Stărică, C., (1999). Change of structure in financial time series, long range dependence and the GARCH model. Preprint.

Miller, M.H., (1991). Financial Markets and Market Volatility. Basil Blackwell, Cambridge, MA.

Newey, W.K., West, K.D., (1987). A Simple Positive Definite, Heteroskedasticity and Autocorrelation Consistent Covariance Matrix. Econometrica 55, 703–708.

Paparoditis, E., Politis, D., (2003). Residual–based block bootstrap for unit root testing. Econometrica 71, 813–855.

Perron, P., (1989). The great crash, the oil price shock, and the unit root hypothesis. Econometrica 57, 1361–1401.

Phillips, P.C.B., Perron, P., (1988). Testing for a unit root in time series regression. Biometrika 75, 335–346.

Rachev, S.T., Mittnik., S., (2000). Stable Paretian Models in Finance. John Wiley & Sons, New-York.

Račkauskas, A., Suquet, C., (2003). Hölderian invariance principle and some applications for testing epidemic changes, in: Teyssière, G., Kirman, A. (Eds.), Long–Memory in Economics. Springer Verlag, Berlin, forthcoming.

Robinson, P.M., (2001). The memory of stochastic volatility models. Journal of Econometrics 101, 195–218.

Robinson, P.M., (1995). Gaussian semiparametric estimation of long range dependence. The Annals of Statistics 23, 1630–1661.

Robinson, P.M., (1994). Time series with strong dependence, in: Sims, C.A. (Ed.) Advances in Econometrics, Sixth World Congress. Cambridge University Press, pp. 47–95.

Robinson, P.M., (1991). Testing for strong serial correlation and dynamic conditional heteroskedasticity in multiple regression. Journal of Econometrics 47, 67–84.

Shiller, R.J., (2000). Irrational Exuberance. Princeton University Press, Princeton, N.J.

Shiller, R.J., (1981). Do stock prices move by too much to be justified by subsequent changes in dividends? American Economic Review 71, 421–436.

Stărică, C., Granger, C.W.J., (2001). Non–stationarities in stock returns. Review of Economics and Statistics, forthcoming.

Stiglitz, J.E., (1990). Symposium on bubbles. Journal of Economic Perspectives 4, 13–18.

Teyssière, G., (2003). Interaction models for common long–range dependence in asset price volatility, in: Rangarajan, G., Ding, M. (Eds.), Processes with Long Range Correlations: Theory and Applications, Lecture Notes in Physics, Vol. 621. Springer Verlag, Berlin, pp. 251–269.

Teyssière, G. (1998). Multivariate long–memory ARCH modelling for high frequency foreign exchange rates, in: Proceedings of the HFDF-II Conference, Zurich 1998. Olsen & Associates, Zurich.

Teyssière, G., (1997). Modelling exchange rates volatility with multivariate long–memory ARCH processes. Under revision for the Journal of Business and Economic Statistics.

Teyssière, G., (1996). Double long–memory financial time series. Preprint.

Theil, H., (1961). Economic Forecast and Policy. North Holland, Amsterdam.

Tirole, J., (1985). Asset bubbles and overlapping generations. Econometrica 53, 1499–1528.

Townsend, R.M., (1983). Forecasting the forecasts of others. Journal of Political Economy 91, 546–588.

Veitch, D., Abry, P., (1999). A wavelet based joint estimator of the parameters of long–range dependence. IEEE Transactions on Information Theory 45, 878–897.

Vostrikova, L.Ju., (1981). Detection of "disorder" in multidimensional random processes. Soviet Mathematics Doklady 24, 55–59.

Wärneryd, K–E., (2001). Stock Market Psychology. Edwar Elgar, Cheltenham, UK.

West, K.D., (1988). Bubbles, fads and stock price volatility tests: a partial evaluation. Journal of Finance 43, 639–656.

West, K.D., (1987). A specification test for speculative bubbles. Quarterly Journal of Economics 102, 553–580.

Woo, W.T., (1987). Some evidence of speculative bubbles in the foreign exchange markets. Journal of Money, Credit and Banking 19, 499–514.

Wu, G., Xiao, Z., (2002). Are there speculative bubbles in stock markets? Evidence from an alternative approach. Preprint.

Xiao, Z., Phillips, P.C.B., (2002). A CUSUM test for cointegration using regression residuals. Journal of Econometrics 108, 43–61.

Yao, Y., (1988). Estimating the number of change–points via Schwarz criterion. Statistics and Probability Letters 6, 181–189.

Zarepour, M., Knight, K., (1991). Bootstrapping unstable first order autoregressive process with errors in the domain of attraction of a stable law. Communications in Statistics-Stochastic Models 15, 11–27.

A Tables

Table 1: Bubbles Tests. (Test size 5%)

Model	P–P	K–P	J–K	W–X$_{M_1}$	W–X$_{M_2}$	W–X$_{M_3}$
Without memory	0.8380	0.9542	0.6029	0.9911	0.9507	0.8716
With memory	0.8692	0.9680	0.6947	0.9971	0.9651	0.8982

- P–P: Percentage of rejection of the unit root hypothesis with the test by Paparoditis and Politis (2003),
- K–P: Percentage of rejection of the unit root hypothesis with the test by Kokoszka and Parfionovas (2004),
- J–K: Percentage of rejection of the unit root hypothesis with the test by Jach and Kokoszka (2004),
- W–X: Percentage of rejection of the hypothesis of no bubbles with the test by Wu and Xiao (2002), for different choices of the bandwidth parameter: M_1, M_2, M_3.

Table 2: Change–Point Tests.

Model	CVM	K–L	I–T	Lav.	M–S
Without memory	0.2890	2.9605	4.0300	1.9081	6.9059
With memory	0.2875	2.2346	3.6510	1.7302	5.3871

- CVM: Percentage of rejection of the null hypothesis of no change point, for the Cramér–von Mises test, (Test size 5%),
- K–L: Average number of change–points detected by the test by Kokoszka and Leipus (1999), combined with the binary segmentation procedure,
- I–T: Average number of change–points detected by the test by Inclán and Tiao (1994),
- Lav.: Average number of change–points detected by Lavielle's (1999) test,
- M–S: Average number of change–points detected by the goodness of fit test by Mikosch and Stărică (1999).

Table 3: Estimation of the scaling coefficient for the absolute returns and squared returns.

Model	Local Whittle $\|R_t\|$	Wavelets $\|R_t\|$	Local Whittle R_t^2	Wavelets R_t^2
Without memory	0.6270	0.1389	0.5898	0.1456
With memory	0.5868	0.0875	0.5678	0.0914

B Graphs

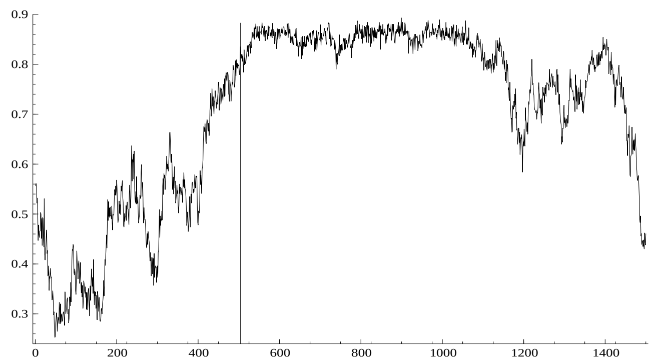

Figure 1: A realisation of the process $\{w_t\}$. The vertical line shows the location of the change–point detected by the CUSUM test of Kokoszka and Leipus (2000).

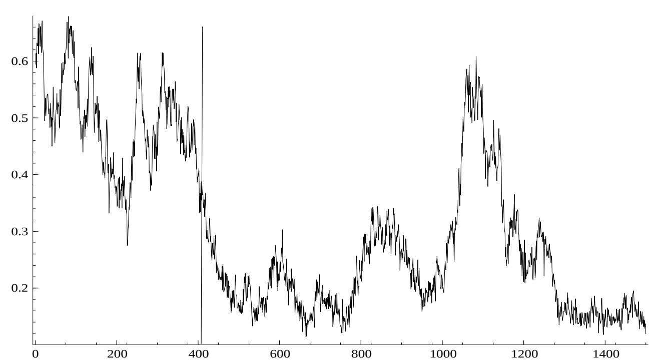

Figure 2: A realisation of the process $\{w_t\}$. The vertical line shows the location of the change–point detected by the CUSUM test of Kokoszka and Leipus (2000).

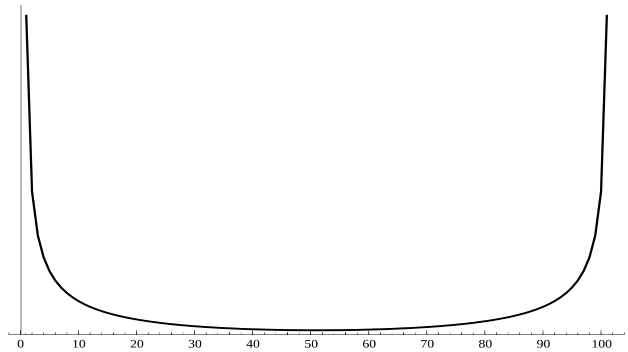

Figure 3: $\mu(k)$ with $\varepsilon = 0.005$, $\delta = 0.01$, $N = 100$.